P.G. Demidov Yaroslavl State University
Regional Educational & Research Mathematical Centre
"Centre of Integrable Systems"
Steklov Mathematical Institute, RAS, Moscow

International Scientific Conference

INTEGRABLE SYSTEMS AND NONLINEAR DYNAMICS

ABSTRACTS

Yaroslavl, October 1–5, 2018

Ярославский государственный университет
им. П.Г. Демидова
Региональный научно-образовательный математический центр
"Центр интегрируемых систем"
Математический институт им. В.А. Стеклова, РАН, Москва

Международная научная конференция

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ON DAMPED CLASSICAL OSCILLATIONS IN THE MODIFIED PÖSCHL-TELLER POTENTIAL WELL

Alekseeva E. S.[1], Rassadin A. E.[2]

[1]*Independent University of Moscow, Russia;* kometarella@mail.ru
[2]*Nizhny Novgorod Mathematical Society, Nizhny Novgorod, Russia;* brat_ras@list.ru

This report continues our investigations in the framework of research program claimed in [1] namely we shall find here asymptotic solution of the next Cauchy problem under $0 < \delta << 1$:

$$\ddot{x} + 2\,\delta\,\dot{x} + \frac{2\,\sinh x}{\cosh^3 x} = 0\,, \quad x(0) = 0\,, \quad \dot{x}(0) = p_0\,. \tag{1}$$

Let us rewrite equation (1) as quasi-Hamilton system [2]:

$$\dot{x} = \frac{\partial H(x,p)}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H(x,p)}{\partial x} - 2\,\delta\,p\,, \tag{2}$$

with Hamiltonian

$$H(x,p) = \frac{p^2}{2} - \frac{1}{\cosh^2 x}\,. \tag{3}$$

Pöschl and Teller were the first who considered the potential well $U(x) = -\cosh^{-2} x$ for needs of quantum mechanics [3]. 30 years later such potential in dimensional variables arose as reflectionless potential in the theory of the Korteweg-de Vries equation (see [4] and references therein).

Further let us do in system (2) the following symplectic change of variables:

$$\sinh x(I,\theta) = \frac{\sqrt{I\,(\sqrt{8}-I)}}{\sqrt{2}-I}\,\sin\theta, \quad p(I,\theta) = \frac{(\sqrt{2}-I)\,\sqrt{I\,(\sqrt{8}-I)}\,\cos\theta}{2\,\sin^2\theta + (\sqrt{2}-I)^2\,\cos^2\theta}\,. \tag{4}$$

The result of substitution (4) into system (2) is equal to:

$$\dot{I} = -\frac{2\,\delta\,(\sqrt{2}-I)\,I\,(\sqrt{8}-I)}{2\,\sin^2\theta + (\sqrt{2}-I)^2\,\cos^2\theta}, \quad \dot{\theta} = \sqrt{2} - I + \frac{2\,\delta\,\sin 2\,\theta}{2\,\sin^2\theta + (\sqrt{2}-I)^2\,\cos^2\theta}\,. \tag{5}$$

Formulae (4) are exact solution of equation (1) under $\delta = 0$, I and θ being the action-angle variables for unperturbed finite motion [5].

Because of δ is small parameter then according to approach developed in [2] one can apply period averaging to system (5). Thus having done this operation instead of system (5) we obtain the following truncated system:

$$\dot{I} = -2\,\delta\,I, \qquad \dot{\theta} = \sqrt{2} - I. \tag{6}$$

Straightforward integration of system (6) gives:

$$I_{as}(t) = I_0\,\exp(-2\,\delta\,t)\,, \qquad \theta_{as}(t) = \sqrt{2}\,t - I_0\,\frac{1-\exp(-2\,\delta\,t)}{2\,\delta}\,. \tag{7}$$

In correspondence with general method suggested in [1] in order to find asymptotic solution of system (2) or input equation (1) it is necessary to insert formulae (7) into expressions (4):

$$x_{as}(t) = x(I_{as}(t), \theta_{as}(t)), \qquad p_{as}(t) = p(I_{as}(t), \theta_{as}(t)). \tag{8}$$

In the report presented comparison of accuracy of asymptotic solution (8) of system (2) with its numerical solution is carried out under different values of δ and initial action variable I_0. Moreover the same procedure is performed for expressions (7) and numerical solution of system (5). Hence if global accuracies of numerical methods for solving both system (2) and system (5) are known then vector of total error ($|x(t)-x_{as}(t)|$, $|p(t)-p_{as}(t)|$, $|I(t)-I_{as}(t)|$, $|\theta(t)-\theta_{as}(t)|$) can be estimated too.

On the basis of above described result asymptotic solutions of the next damped Klein-Gordon-Fock equation:

$$\frac{\partial^2 v}{\partial t^2} + 2\,\delta\,\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + \frac{2\,\sinh v}{\cosh^3 v} = 0 \tag{9}$$

and the following reaction-diffusion equation:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{2\,\sinh v}{\cosh^3 v} \tag{10}$$

can be constructed because of equations (9) and (10) are reduced to input ordinary differential equation (1) by means of automodelling substitutions namely for equation (9) one ought to use substitution $v(x,t) = V\left(\frac{\omega\,t - k\,x}{\sqrt{\omega^2 - k^2}}\right)$ with $\omega > k > 0$ and for equation (10) substitution $v(x,t) = V(x - c\,t)$ with small $c > 0$ is required.

In conclusion it is necessary to note that in quantum mechanics in the framework of method of factorization the modified Pöschl-Teller

Hamiltonian (3) can be presented as hierarchy of creation-annihilation operators [6] therefore in accordance with algorithm proposed in [7] our results may be useful under description of behaviour of quantum particle in the modified Pöschl-Teller potential well under taking into account quantum mechanical dissipation.

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CLASSICAL DYNAMICS ON A GRAPH AND SEMICLASSICAL EIGENFUNCTIONS OF THE SCHRÖDINGER OPERATOR THAT ARE LOCALIZED NEAR A SUBGRAPH

Allilueva A. I.

Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia; esina_anna@list.ru

Semiclassical asymptotics of eigenvalues and eigenfunctions of quantum operators can be associated with invariant geometrical objects of corresponding classical Hamiltonian systems. We study semiclassical eigenfunctions for the Schrödinger operator on a metric graph that are localized near certain subgraph. Localization is provided by special properties of the corresponding classical dynamics. Namely, the motion outside the subgraph must be blocked either by jumps of the potential or by non-trivial boundary conditions in vertices.

INVESTIGATION OF FAMILIES OF SYMMETRIC PERIODIC SOLUTIONS OF HILL PROBLEM AND ITS GENERALIZATION

Batkhin A. B.

Keldysh Institute of Applied Mathematics of RAS, Moscow, Russia;
Moscow Institute of Physics and Technology, Dolgoprudny, Russia;
batkhin@gmail.com

We consider nonintegrable Hamiltonian system with two degrees of freedom, namely well-known celestial mechanics *Hill problem*, as a singular perturbation of an integrable one with quadratic Hamiltonian. Hill problem describes the motion of a massless body near the minor of two active masses and it is widely used in celestial mechanics and cosmodynamics.

Technique of generating solutions (see [1, 2] for the case of the restricted three-body problem) is applied for studying families of symmetric periodic orbits. Each one-parameter family of such solution is described in terms of a sequence of so called *arc-solutions*, conjugated to each other over the certain rules by hyperbolic conics. The arc-solutions are such solutions of the integrable unperturbed problem that start and finish at the origin – the singular point of the perturbation function. Such approach allows describing not only periodic orbits but any invariant structure of the dynamical system that can be continued up to the limiting integrable problem as well.

Using generating solution it is possible to predict such properties of corresponding family of periodic orbits as a type of symmetry, global multiplicity of the orbit of generated solution and first approximation of the initial conditions and the period of the solution. An algorithm for investigation of families of symmetric periodic solutions over its generating sequence was proposed in [3]. This algorithm is applied to finding families of symmetric periodic obits of Hill problem. More than fifty new families of periodic solutions with different types of symmetry were found out and completely investigated [4]. The symmetry of generating solution plays an essential role for obtaining the initial condition of periodic orbit of the family. For computation of the whole family from one periodic solution a kind of predictor-corrector method is applied, which essentially explores the structure of the monodromy matrix of the periodic solution and provides the monitoring of bifurcations of the family. The obtained with numerical computations data of symmetric periodic solutions allows to predict some properties of the whole family of periodic orbits.

Some generalization of the original Hill problem, which includes the singular perturbation with the opposite sign, was considered as well. Hamiltonian function of the generalized Hill problem takes the following form

$$H = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 - x_1^2 + \frac{1}{2} x_2^2 + \frac{\sigma}{r},$$

where $r = \sqrt{x_1^2 + x^2}$ and $\sigma = \pm 1$. For value $\sigma = -1$ one gets the Hamiltonian of the classical case of Hill problem. We call the problem with $\sigma = +1$ as *anti-Hill problem*. For value $\sigma = 0$ one gets so called integrable *Hénon problem* (known as ClohessyвЂ“Wiltshire equations) which particular solutions are used for construction generating sequences of families of periodic orbits of the generalized problem. The structure of families of periodic solutions of anti-Hill problem is considerably simpler than in the case of Hill problem and could be totally described with their generating solutions.

Intensive numerical computations allow to state that all known families of periodic solutions of Hill problem can be continued into the families of periodic solutions of anti-Hill problem but not vice verse, i. e. there are some anti-Hill problem's families that cannot be continued into Hill problem ones. More over, the further numerical experiment demonstrated that all families of Hill and anti-Hill problems form the common network connecting to each other by common generating solutions and by sharing common orbits with integer multiplicity of different families as well [5].

In the present work we provide an improved version of the algorithm based on the asymptotic of the family of periodic orbits. With the help of this algorithm, further computations of families of periodic solutions were carried out. These computations allow us to formulate some hypotheses about the connection between the structure of the generating solution of the family and some properties of its periodic solutions.

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ATTRACTORS AND BIFURCATIONS IN CERTAIN SYSTEMS

Belykh V. N.

Volga state university of water transport, Nizhny Novgorod, Russia;
Nizhny Novgorod state university, Nizhny Novgorod, Russia;
belykh@unn.ru

In this talk we present the auxiliary $2d$ systems method for bifurcational analysis of multidimensional system with one nonlinearity. This method is the main tool of rigorous proof of the homoclinic and heteroclinic linkages in dynamical systems [1,2]. This method is illustrated for different systems with two or three equilibrium points, namely Phase Locked Loop system, Chua system, Anischenko-Astakhov system, etc.

Homoclinic orbits of saddle-focus leading to emergence of chaotic attractors are presented. An heuristic model of new attractor corresponding to complicated limiting set similar to Ressler attractor in a system with two equilibria is discussed.

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ON POLYHEDRAL GRAPHS OF BALANCED AND UNBALANCED BIPARTITE SUBGRAPH PROBLEMS

Bondarenko V. A.[1], Nikolaev A. V.[2], Shovgenov D. A.[3]

[1]*P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
bond@bond.edu.yar.ru
[2]*P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
andrei.v.nikolaev@gmail.com
[3]*P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
djsh92@mail.ru

We consider three NP-hard problems of constructing an optimal biclique in a bipartite graph [1].

Weighted balanced complete bipartite subgraph. Given a complete bipartite graph $G = (U, V, E)$, $|U| = |V| = n$, a weight function $C : E \to \mathbb{R}$, and a positive integer $k \leq n$. It is required to find a balanced complete bipartite subgraph $G_x = (U_x, V_x, E_x)$ with the largest sum of edge weights such that $|U_x| = |V_x| = k$.

Maximum weighted complete bipartite subgraph. Given a complete bipartite graph $G = (U, V, E)$, $|U| = |V| = n$, a weight function $C : E \to \mathbb{R}_+$, and a positive integer $k \leq 2n$. It is required to find a complete bipartite subgraph $G_x = (U_x, V_x, E_x)$ with the sum of edge weights being as large as possible such that $|U_x| + |V_x| = k$.

Minimum weighted complete bipartite subgraph. Given a complete bipartite graph $G = (U, V, E)$, $|U| = |V| = n$, a weight function $C : E \to \mathbb{R}_+$, and a positive integer $k \leq 2n$. It is required to find a complete bipartite subgraph $G_x = (U_x, V_x, E_x)$ with the sum of edge weights being as small as possible such that $|U_x| + |V_x| = k$.

The object of the research is the construction of cone decomposition. Let X be a finite set of points in $\mathbb{R}^d$. We consider the problem of maximizing a linear objective function over X:

$$\langle c, x \rangle \to \max, \ x \in X.$$

We denote by

$$K(x) = \{c \in \mathbb{R}^d : \ \langle c, x \rangle \geq \langle c, y \rangle \,, \ \forall y \in X\}.$$

Since $K(x)$ is the set of solutions of a finite system of homogeneous linear inequalities, it is a convex polyhedral cone. The set of all cones $K(x)$ is called the *cone decomposition* of the space $\mathbb{R}^d$ by the set X.

We consider the graph of the cone decomposition with the cones being the vertices, and two cones $K(x)$ and $K(y)$ are adjacent if and only if they have a common facet:

$$\dim(K(x) \cap K(y)) = d - 1.$$

We denote by $\omega(X)$ the clique number, the number of vertices in a maximum clique, of the graph of the cone decomposition of the space $\mathbb{R}^d$ by the set X. It is known [2] that the complexity of the direct type algorithms, based on linear comparisons, of finding the maximum (or minimum, if we change the sign of inequality in the definition of cone) of a linear objective function over the set X, or, in other words, finding the cone $K(x)$ with the vector c, cannot be less than the value of $\omega(X) - 1$. Thus, $\omega(X)$ characterizes the time complexity in a broad class of algorithms.

Similarly, we construct the cone decompositions of the positive orthant $\mathbb{R}^d_+$ for the problems on maximum and minimum:

$$K^+_{\max}(x) = \{c \in \mathbb{R}^d_+ : \ \langle c, x \rangle \geq \langle c, y \rangle \ , \ \forall y \in X\},$$
$$K^+_{\min}(x) = \{c \in \mathbb{R}^d_+ : \ \langle c, x \rangle \leq \langle c, y \rangle \ , \ \forall y \in X\}.$$

This construction, in turn, is dual to the polyhedron of the problem that is defined as the dominant of the convex hull of the set X and is used in the analysis of problems with nonnegative input data [3,4,5]. In our case it is the nonnegative weights of edges.

Theorem 1. *The clique number of the graph of the cone decomposition $K_{n,k}$ for the weighted balanced complete bipartite subgraph problem is superpolynomial in n and k:*

$$\omega(K_{n,k}) \geq \binom{n}{k} = \Omega\left(\left(\frac{n}{k}\right)^k\right).$$

Theorem 2. *The clique number of the graph of the cone decomposition $K^{max}_{n,k}$ for the maximum weighted complete bipartite subgraph problem is superpolynomial in n and k:*

$$\omega(K^{max}_{n,k}) \geq \binom{n}{s} = \Omega\left(\left(\frac{n}{s}\right)^s\right), \ \textit{if } k = 2s,$$
$$\omega(K^{max}_{n,k}) \geq \binom{n-1}{s} = \Omega\left(\left(\frac{n-1}{s}\right)^s\right), \ \textit{if } k = 2s+1.$$

Theorem 3. *Let*

$$k = 3s, m = \left\lfloor \frac{n}{2} \right\rfloor, \frac{9}{4}m < k < 3m,$$

then the clique number of the graph of the cone decomposition $K_{n,k}^{min}$ *for the minimum weighted complete bipartite subgraph problem is superpolynomial in* n *and* k*:*

$$\omega(K_{n,k}^{min}) \geq \binom{m}{s} = \Omega\left(\left\lfloor \frac{3n}{2k} \right\rfloor^{\frac{k}{3}}\right).$$

For three considered problems of the balanced subgraphs with arbitrary edges and unbalanced subgraphs of minimum and maximum weight with nonnegative edges we establish the superpolynomial clique numbers of the graphs of the cone decompositions. In all three cases, the polyhedral characteristics correlate with the complexity of the problems.

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ON POLYNOMIAL SOLVABILITY OF INTEGER RECOGNITION PROBLEM

Bondarenko V. A.[1], **Nikolaev A. V.**[2]
[1]*P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
bond@bond.edu.yar.ru
[2]*P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
andrei.v.nikolaev@gmail.com

We consider a problem of integer recognition (IR): for a given linear objective function $f(x)$ and a polytope P determine whether $\max\{f(x) \mid x \in P\}$ is achieved at an integral vertex of P. In a certain sense, IR occupies an intermediate place between the classical problems of linear programming (LP) and integer linear programming (ILP).

The IR problem was considered (apparently for the first time) in [1], where, in particular, it was proved that it is polynomially solvable for the class of the rooted semimetric polytopes $RMET(n)$. The rooted semimetric polytopes are one of the simplest LP relaxations of the Boolean quadratic programming and maximum cut problems. Thus, ILP over $RMET(n)$ is NP-hard [2].

On the other hand, there are classes of polytopes for which IR is NP-hard. In particular, the metric polytope $MET(n)$, defined by triangle inequality constraints [3], and the 3-SAT relaxation polytopes $SAT^{LP}_{m,n}$ [1]. Moreover, IR turns out to be NP-hard over polytopes, obtained by complementing the constraints of $RMET(n)$ by a single linear inequality [4]. Polynomially solvable subproblems of IR over $SAT^{LP}_{m,n}$ are considered in [5]. Based on them, there was developed a polynomial time algorithm for some special cases of edge-constrained bipartite graph coloring problem [5].

The obtained results can be generalized in the form of the following statement: a sufficient condition for the polynomial solvability of the IR problem.

Theorem 1. *Let P be a polytope in $\mathbb{R}^n$, defined by the system of linear constraints $Ax = 1$, $x \geq 1$, where A is a Boolean matrix. Let the polytope Q be embedded in P and defined by a system of integer linear constraints whose number is polynomial in n. Let the integral vertices of P and Q coincide. Finally, we suppose that for any point u of Q there exists an integral vertex z that is majorized by u (for some $a > 0$ we have $az \leq u$). Then IR over P is polynomially solvable.*

The assumption that A is a Boolean matrix is not burdensome — for a large number of the most known problems it is satisfied. A key condition of the theorem is the majorization of integral vertices. It holds for $RMET(n)$ $(= P)$ and $MET(n)$ $(= Q)$, and IR is polynomially solvable over rooted semimetric polytopes [1].

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NEW INTEGRABLE SYSTEMS OF NONHOLONOMIC DYNAMICS

Borisov A. V.[1,2], Kilin A. A.[3], Mamaev I. S.[4]

[1] *A. A. Blagonravov Mechanical Engineering Research Institute of RAS, ul. Bardina 4, Moscow, 117334 Russia;*
[2]*Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi, 141700 Russia;*
borisov@rcd.ru
[3]*Udmurt State University, ul. Universitetskaya 1, Izhevsk, 426034 Russia;* aka@rcd.ru
[4]*Izhevsk State Technical University, ul. Studencheskaya 7, Izhevsk, 426069 Russia;* mamaev@rcd.ru

Equations of motion are obtained for the problem of a homogeneous ball rolling without slipping on a rotating surface. Integrals of motion and an invariant measure are found. A detailed linear stability analysis of the ball’s rotations at the saddle point of a hyperbolic paraboloid is presented. A three-dimensional Poincare map generated by the phase flow of the problem is numerically investigated and the existence of a region of bounded trajectories in a neighborhood of the saddle point of the paraboloid is demonstrated. It is shown that a similar problem of a ball rolling on a rotating paraboloid, considered within the framework of the rubber model [2], can be reduced to a Hamiltonian system which includes the Brower problem [1] as a particular case.

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INTEGRABLE HAMILTONIAN POLYNOMIAL SYSTEMS ON $\mathbb{R}^4$ DEFINED BY HYPERELLIPTIC CURVES

Buchstaber V. M.[1], Mikhailov A. V.[2,3]

[1]*Steklov Mathematical Institute RAS, Moscow, Russia;* buchstab@mi-ras.ru

[2]*School of Mathematics, University of Leeds, Leeds, UK;* a.v.mikhailov@leeds.ac.uk

[3]*Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;* a.v.mikhailov@gmail.com

We construct Lie algebras of vector fields on universal bundles $\mathcal{U}_g$ of symmetric squares of nondegenerate hyperelliptic curves of genus $g = 1, 2, \ldots$ For each of these Lie algebras, the Lie subalgebra of vertical fields has two commuting generators. The Lie algebra of projectable fields is isomorphic to the polynomial Lie algebra of vector fields, that are tangent to the discriminant variety of the universal hyperelliptic curve of genus g. We show that the generators of the Lie subalgebra of projectable fields determine the representation of the Lie subalgebra of the Witt algebra. We explicitly construct a birational isomorphism $\mathbb{C}^{2g+2} \to \mathcal{U}_g$. As an application, we obtain integrable Hamiltonian polynomial dynamical systems with polynomial Hamiltonians on $\mathbb{C}^4$ and $\mathbb{R}^4$. In cases $g = 1, 2, 3$ we explicitly describe the solutions of these systems and discuss their properties.

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TOPOLOGICAL RECURSION FOR BOUSQUET-MELOU–SHAEFFER NUMBERS

Bychkov B. S.

HSE,CIS,MIPT, Moscow-Yaroslavl, Russia;
boris.bychkov@gmail.com

Topological recursion is a recurent procedure that is used to find elements of generating series; generating series coming from enumerative geometry in particular. I will talk about first example of so-called global topological recursion: a recursion for Bousquet-Melou–Shaeffer numbers which counts ramified coverings of the sphere with fixed quantity of ramification points and fixed overal degeneracy.

ASYMPTOTICALLY LYAPUNOV-STABLE SOLUTIONS WITH BOUNDARY AND INTERNAL LAYERS IN THE STATIONARY REACTION-DIFFUSION-ADVECTION PROBLEMS WITH A SMALL TRANSFER

Davydova M. A.[1], Nefedov N. N.[2], Zakharova S. A.[3]

[1]*Moscow State University, Moscow, Russia;*
m.davydova@physics.msu.ru
[2]*Moscow State University, Moscow, Russia;* nefedov@phys.msu.ru
[3]*Moscow State University, Moscow, Russia;*
sa.zakharova@physics.msu.ru

The stationary distribution of the gas impurity concentration $u(x)$ in the dimensionless variables is described by the problem for the stationary equation of the reaction-diffusion-advection type:

$$\begin{aligned}&\nabla(k(x)\nabla u) - \nabla(u\mathbf{A}) + f(u,x) = 0, \quad x \in D \subset R^3,\\ &-k(x)(\nabla u, \mathbf{n}) = G(x), \quad x \in S,\end{aligned}$$

where $\mathbf{A}(x)$ is a velocity of transfer, $f(u,x)$ is a volume density of sources (or sinks) of matter, $G(x)$ is a flow of matter at the boundary S, $\mathbf{n}$ is a unit external normal to the S at the point x. The turbulent diffusion coefficient $k(x)$ is a small function.

In the approximation of an incompressible medium, under the certain physical state of the atmospheric surface layer, the concentration distribution of the gas impurity in the atmospheric surface layer is described by the stationary problem:

$$\begin{aligned} &\varepsilon^2\Delta u - \varepsilon(\mathbf{A}(x), \nabla u) - B(u,x) = 0, \quad x = (x_1, x_2) \in D \subset R^2, \\ &\frac{\partial u}{\partial n}\Big|_S = g(x), \quad x \in S, \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is a small parameter, the components of the vector $\mathbf{A}(x) = \{A_1(x), A_2(x)\}$, the functions $B(u,x)$, $g(x)$ and the boundary S are assumed to be sufficiently smooth. We negelect the dependence of the coefficient $k(x)$ on the vertical variable.

The problem (1), modeling the processes of the transport, decay and chemical transformation of the active and passive impurities in the surface layer of the atmospher, is investigated by use of the methods of asymptotic analysis [1-4], developing the classical methods [5] for the case of the stationary multidimensional reaction-diffusion-advection problems.

We study the asymptotically Lyapunov-stable solutions of problem (1) of the boundary layer type and the contrast structures by constructing the formal asymptotic approximations of an arbitrary-order accuracy based on the boundary-function method. To justify the constructed asymptotics, we use an asymptotic method of differential inequalities. The results of the study are illustrated by the example of the two-dimensional boundary value problem with a cubic nonlinearity. They can be used to create a numerical algorithm that uses asymptotic analysis to construct spatially inhomogeneous mashes when describing the internal layer of contrast structure, and also for the purposes of constructing the test examples.

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DELONE SETS AND CRYSTALLINE STRUCTURE

Dolbilin N. P.

Steklov Mathematical Institute, Moscow, Russia;
dolbilin@mi.ras.ru

In this talk, we discuss a problem posed in the Delone group at the Steklov Mathematical Institute in the 1970s. The problem is about genesis of the global order in the atomic structure of crystals. A very high symmetry of the atomic structure of crystalline materials appears from amorphous state of melt, solution, or gas during crystallization. One of the core problems of the *local theory for regular systems* was – and still is– to explain the genesis of symmetry of a crystalline structure through congruence of its relatively small by size fragments/clusters.

An ideal crystal is defined (after E. Fedorov) as a Delone set X that is a union of several $\mathrm{Sym}(X)$-orbits, where $\mathrm{Sym}(X)$ is the orthogonal group of symmetries of X. Each individual orbit $\mathrm{Sym}(X)\cdot x$, $x \in \mathbb{R}^d$, is a Delone set on that the group $\mathrm{Sym}(X)$ acts point-transitively. Such a point set with a transitive group is called a *regular system.*

For a given $x \in X$ and $\rho > 0$ a subset of all points $x' \in X$ with $|xx'| \le \rho$ is called a *ρ-cluster* $C_x(\rho)$ of the point x. Given x and $y \in X$, ρ-clusters $C_x(\rho)$ and $C_y(\rho)$ are said to be *equivalent* if there is an isometry g such that $g(x) = x'$ and $g(C_x(\rho)) = C_x(\rho)$. For given ρ the number of all classes of ρ-clusters in X is supposed to be finite and denoted by $N(\rho)$. It is obvious that the value $N(\rho)$ does not decrease as the ρ increases.

It is easy to see that X is a regular system if for all $\rho > 0$ $N(\rho) = 1$. Assume now that for X and for some ρ_0 the function . Is it sufficient for

X to be a regular system? From the local theory it follows that there is such a ρ_0 that from the assumption $N(\rho_0) = 1$ it follows that X is regular.

For given dimension d the *regularity radius* $\hat{\rho}_d$ is defined by the two conditions: (1) $N(\hat{\rho}_d) = 1$ implies regularity of the $X \subset \mathbb{R}^d$, and (2) for $\forall\varepsilon > 0$ there is a non-regular Delone set X with $N(\hat{\rho}_d - \varepsilon) = 1$.

As proved recently, $\hat{\rho}_d \geq d2R$, where R is the parameter of a Delone set $X \subset \mathbb{R}^d$ (a radius of the biggest 'empty' ball, i.e. a ball containing no points from X).

In the talk we will discuss the case $d = 3$ and show that $6R \leq \hat{\rho}_3 \leq 10R$. In fact, the upper bound for $\hat{\rho}_3$ is based on the symmetry group $S_x(2R)$ of the $2R$-cluster $C_x(2R)$ in X. We emphasize that finite subgroups of $O(3)$, that can be realized as a group $S_x(2R)$ of the $2R$-cluster in X with condition $N(2R) = 1$, constitute a finite list L. It is particularly interesting that for many groups from the list L, it turns out that in order to provide regularity of a Delone set, it is sufficient to require condition $N(2R) = 1$ which is about the minimal condition.

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QUASI-STABLE STRUCTURES IN CIRCULAR CHAINS OF UNIDIRECTIONALLY COUPLED OSCILLATORS

Glyzin S. D.[1], Kolesov A. Yu.[2], Rozov N. Kh.[3]

[1]*P.G. Demidov Yaroslavl State University, Russia;*
glyzin.s@gmail.com
[2]*P.G. Demidov Yaroslavl State University, Russia;*
kolesov@uniyar.ac.ru
[3]*M.V. Lomonosov Moscow State University, Russia;*
fpo.mgu@mail.ru

Artificial genetic oscillators are very interesting because they are simple models of key biological processes such as the cell cycle and circadian rhythms. The simplest genetic oscillator, proposed in [1] and called a repressilator, consists of three elements. Each element unidirectionally inhibits a neighboring element. The mathematical model of this gene network has the form

$$\begin{gathered}\dot m_j = -m_j + \alpha/(1+u_{j-1}^{\gamma})^{-1} + \alpha_0,\ \dot u_j = \beta(m_j - u_j),\\ j = 1,2,3,\ u_0 = u_3.\end{gathered} \tag{1}$$

We assume that each oscillator element is a set of mRNA (matrix ribonucleic acid) of concentration m_j and protein of concentration u_j. We further assume that the temporal variation of m_j is characterized by processes of synthesis and degradation. The first process is described by the function $\alpha/(1+u_{j-1}^{\gamma})$, where u_{j-1} is the concentration of the repressor protein for the jth mRNA, $\gamma = \mathrm{const} > 0$ is the cooperativity coefficient, and $\alpha = \mathrm{const} > 0$ is the transcription rate in the absence of the repressor. The second process is described by the term $-m_j$. And finally, the constant $\alpha_0 > 0$ is responsible for the promoter leakage. The protein dynamics is represented by linear processes of synthesis (the term βm_j in the system (1)) and degradation (the term $-\beta u_j$). Here, $\beta = \mathrm{const} > 0$ is the ratio of the protein degradation rate to the mRNA degradation rate.

As a rule, model (1) is studied under the assumption that β and α_0 are small. In this situation, changing the variables $\beta t \to t$ and omitting the additional term α_0, we obtain a singularly perturbed system and then apply the well-known Tikhonov reduction principle to it. The problem of self-oscillations of similar systems arising in modeling gene networks has been studied by many authors (see, e.g., [2]–[7]).

The character of the interaction between the concentrations u_j and m_j described above is quite similar to the interaction between six ecological populations comprising three predators and three prey. Indeed, we assume that u_j and m_j, $j = 1, 2, 3$, are the respective densities of the population sizes of the predators and their prey. By (1), each predator u_j then feeds only on one prey m_j (for $m_j \equiv 0$, the magnitude u_j decays according to an exponential law) and simultaneously exerts pressure on only the prey m_{j+1}. The latter is manifested by the rate of increase in m_{j+1} decreasing as u_j increases. Moreover, in the absence of any predator/repressor ($u_{j-1} \equiv 0$), m_j tends to the threshold value $m_j = \alpha + \alpha_0$ as $t \to +\infty$. The above ecological interpretation allows using the Yu.S. Kolesov approach to mathematically model the gene network that interests us. This approach leads to the system

$$\dot{m}_j = r_1(1+a)^{-1}\left[1+a(1-u_{j-1})-m_j\right]m_j+\alpha,\ \dot{u}_j = r_2[m_j-u_j]u_j, \quad (2)$$

$j = 1, 2, 3$, where $u_0 = u_3$ and all constants r_1, r_2, a, and α are positive. If we take the number of oscillators equals to n, then in (2) $j = 1, \ldots, n$ and $u_0 = u_n$. We note that we purposefully add the last term $+\alpha$, which is similar to the last term α_0 in (1), in the equation for m_j and thus break its Volterra structure. Like system (1), the new mathematical model (2) of a repressilator admits a certain simplification. First, we assume that $r_2 \gg 1$, and $r_1 = r \sim 1$. According to the Tikhonov reduction principle, we then have $m_j = u_j$, $j = 1, , \ldots, n$ as $r_2 \to +\infty$ and obtain the system for u_j,

$$\dot{u}_j = r(1+a)^{-1}\left[1+a(1-u_{j-1})-u_j\right]u_j+\alpha, \quad j = 1, \ldots, n, \quad u_0 = u_n. \quad (3)$$

By traveling waves of system (3), we mean special periodic solutions (see [8,9]) that can be represented in the form

$$u_j = u(t+(j-1)\Delta), \quad j = 1, 2, \ldots, n, \quad \Delta = \text{const} > 0. \quad (4)$$

Below, the existence and stability of such solutions are analyzed in the case where $r \gg 1$, $\alpha \ll 1$ and the parameter a is on the order of unity. More precisely, we assume throughout that

$$a = \text{const} > 1, \quad \alpha = r\exp(-b\,r), \quad r \gg 1, \quad b = \text{const} > 0. \quad (5)$$

We proof that, under conditions (5), the number of coexisting periodic solutions (4) to system (3) increases indefinitely as $r \to +\infty$ and $n \to$

$+\infty$ consistently. However, all of them (except for a single stable solution for odd n) are quasi-stable. Namely, the stability spectrum of each of these periodic solutions contains a nonempty group of multipliers $\nu \in \mathbb{C}$, $\nu \neq 1$ lying at a distance of order $\exp(-c\,r)$, $c = \text{const} > 0$ from the unit circle.

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INFINITE-DIMENSIONAL LIE ALGEBRAS ASSOCIATED WITH PARTIAL DIFFERENTIAL EQUATIONS AND BÄCKLUND TRANSFORMATIONS

Igonin S. A.

Centre of Integrable Systems, Yaroslavl State University, Yaroslavl, Russia; s-igonin@yandex.ru

We develop algebraic and geometric methods in the theory of partial differential equations (PDEs). Recall that a distribution on a manifold is a subbundle of the tangent bundle of the manifold. It is known that, using jet bundles, one can regard PDEs as geometric objects. Namely, a PDE can be regarded as a manifold with a distribution (the Cartan distribution) such that solutions of the PDE correspond to certain integral submanifolds of the distribution. This approach is applicable to PDEs satisfying some non-degeneracy conditions, which are almost always satisfied in practice.

Recall that fundamental groups are an important invariant for topological spaces. Using the above-mentioned geometric approach to PDEs, we introduce an analog of fundamental groups for PDEs. However, the "fundamental group of a PDE" is not a group, but a certain system of Lie algebras, which we call fundamental Lie algebras. So, for any given PDE satisfying the above-mentioned non-degeneracy conditions, we define a family of Lie algebras, which we call the fundamental Lie algebras of this PDE. The main idea of the definition is presented below.

Fundamental Lie algebras are new geometric invariants for PDEs and are closely related to integrability properties of PDEs, where integrability is understood in the sense of soliton theory.

The construction of the fundamental Lie algebras for a given PDE is coordinate-independent and is functorial. It gives a functor from the category of PDEs (with a marked point) to the category of Lie algebras.

Before describing this construction for arbitrary PDEs, we would like to outline some applications in the case of (1+1)-dimensional PDEs. Using fundamental Lie algebras, we obtain necessary conditions for integrability and necessary conditions for existence of Bäcklund transformations for (1+1)-dimensional evolution PDEs. For such PDEs, representations of fundamental Lie algebras classify all Lax representations (zero-curvature representations) up to local gauge equivalence.

In the structure of fundamental Lie algebras for integrable (1+1)-dimensional PDEs, one finds infinite-dimensional subalgebras of Kac-Moody algebras and infinite-dimensional Lie algebras of certain matrix-valued functions on some algebraic curves. Considered examples include the Korteweg-de Vries (KdV), Krichever-Novikov, Kaup-Kupershmidt, Sawada-Kotera, nonlinear Schrödinger, Landau-Lifshitz equations.

In a joint work with G. Manno [1] we show that fundamental Lie algebras help to obtain necessary conditions for existence of a Bäcklund transformation (BT) between two given (1+1)-dimensional evolution PDEs. This allows us to prove a number of non-existence results for BTs. For instance, a result of this kind is presented in Theorem 1 below, which is proved in [1].

To describe the main idea of the definition of fundamental Lie algebras for arbitrary PDEs (satisfying the above-mentioned non-degeneracy conditions), we need to recall some notions from topology and category theory.

Let M be a finite-dimensional manifold and $\mathbf{C}(M)$ be the category of topological coverings over M. Let $b \in M$. Consider the *fiber functor* F_b from $\mathbf{C}(M)$ to the category of sets. The functor F_b takes a covering $\tau\colon \tilde{M} \to M$ to the fiber $\tau^{-1}(b)$.

An *automorphism* of a functor F is a natural transformation from F to F such that this transformation is an isomorphism.

Recall that the fundamental group $\pi_1(M, b)$ acts naturally on the fiber $\tau^{-1}(b)$ of every covering $\tau\colon \tilde{M} \to M$. It is known that this gives an isomorphism between $\pi_1(M, b)$ and the group of automorphisms of the fiber functor F_b.

One can make a somewhat similar construction for PDEs as follows. As said above, a PDE can be regarded as a manifold $\mathcal{E}$ with a distribution (the Cartan distribution). There is a notion of coverings of PDEs [3, 6]. Fibers of coverings of PDEs are smooth manifolds. We consider only coverings with finite-dimensional fibers. For each $a \in \mathcal{E}$, this allows us to speak about infinitesimal automorphisms of the corresponding fiber functor F_a, which are defined in terms of vector fields on fibers over a.

Then the fundamental Lie algebra of a PDE $\mathcal{E}$ at a point $a \in \mathcal{E}$ can be defined as the Lie algebra of infinitesimal automorphisms of the corresponding fiber functor F_a. More precisely, one needs to work on the formal level, where smooth functions are replaced by formal power series. Details of this theory will appear in [2].

For (1+1)-dimensional PDEs, fundamental Lie algebras can be described in terms of generators and relations, using a generalization of the Wahlquist-Estabrook prolongation method and gauge transformations.

For any constants $e_1, e_2, e_3 \in \mathbb{C}$, consider the Krichever-Novikov equation

$$\mathrm{KN}(e_1, e_2, e_3) = \left\{ u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{(u-e_1)(u-e_2)(u-e_3)}{u_x}, \quad u = u(x,t) \right\}$$

and the algebraic curve

$$\mathrm{C}(e_1, e_2, e_3) = \left\{ (z, y) \in \mathbb{C}^2 \;\middle|\; y^2 = (z-e_1)(z-e_2)(z-e_3) \right\}.$$

If $e_i \neq e_j$ for all $i \neq j$ then the curve $\mathrm{C}(e_1, e_2, e_3)$ is elliptic.

Consider also the KdV equation $u_t = u_{xxx} + uu_x,\ \ u = u(x,t)$.

Theorem 1. *Let* $e_1, e_2, e_3, e'_1, e'_2, e'_3 \in \mathbb{C}$ *such that* $e_i \neq e_j$ *and* $e'_i \neq e'_j$ *for all* $i \neq j$. *If the curve* $\mathrm{C}(e_1, e_2, e_3)$ *is not birationally equivalent to the curve* $\mathrm{C}(e'_1, e'_2, e'_3)$, *then the equation* $\mathrm{KN}(e_1, e_2, e_3)$ *is not connected with the equation* $\mathrm{KN}(e'_1, e'_2, e'_3)$ *by any Bäcklund transformation* (*BT*).

Also, if $e_1 \neq e_2 \neq e_3 \neq e_1$, *then* $\mathrm{KN}(e_1, e_2, e_3)$ *is not connected with the KdV equation by any BT.*

BTs of Miura type (differential substitutions) for the Krichever-Novikov equation $\mathrm{KN}(e_1, e_2, e_3)$ were studied in [4, 5]. According to [4, 5], the equation $\mathrm{KN}(e_1, e_2, e_3)$ is connected with the KdV equation by a BT of Miura type iff $e_i = e_j$ for some $i \neq j$.

Theorem 1 considers the most general class of BTs, which is much larger than the class of BTs of Miura type studied in [4, 5].

If $e_1 \neq e_2 \neq e_3 \neq e_1$ and $e'_1 \neq e'_2 \neq e'_3 \neq e'_1$, the curves $\mathrm{C}(e_1, e_2, e_3)$ and $\mathrm{C}(e'_1, e'_2, e'_3)$ are elliptic. The theory of elliptic curves allows one to determine when $\mathrm{C}(e_1, e_2, e_3)$ is not birationally equivalent to $\mathrm{C}(e'_1, e'_2, e'_3)$. One gets a certain algebraic condition on the numbers e_1, e_2, e_3, e'_1, e'_2, e'_3, which allows us to formulate the result of Theorem 1 more explicitly. See [1] for details.

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TOPOLOGIAL BILLIARDS BOUNDED WITH ARCS OF CONFOCAL QUADRICS ON MINKOWSKI PLANE IN TERMS OF FOMENKO-ZIESCHANG INVARIANTS

Karginova E. E.

Lomonosov Moscow State University, Moscow, Russia;
karginov13@gmail.com

Minkowski plane is $\mathcal{R}^2$ with scalar product $\langle x, y\rangle = x_1y_1 - x_2y_2$.

The family of confocal quadrics on the Minkowski plane is given by

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b+\lambda} = 1; \qquad \lambda \in \mathcal{R} \qquad (1)$$

Here $a > b > 0$ are real numbers.

The billiard in ellipse on Minkowski plane was investigated by V. Dragovich and M. Radnovich in [2]. They described the topological type of Liouville foliation of this system with the Fomenko-Zieschang method. This method involves obtaining a graph with numeric marks - an invariant of integrable system, which completely determines the topological type of its Liouville foliation. This approach is expanded in [1].

Let us define a simple billiard as a compact, connected subset of a plane which boundary consists of arcs of confocal quadrics of the family (1) and does not contain angles greater than $\frac{\pi}{2}$.

V. V. Fokicheva in [3] proposed and investigated the construction of topological billiard - a 2-dimensional, connected, orientable manifold, obtained as a result of isometric gluing of several simple billiards (sheets) along some convex or straight arcs of the boundary lying on the same quadric.

Let us define the billiard reflection in simple billiard as follows. When the material point hits the boundary arc, it reflects according to the next rule:

- (v_1, v_2) are coordinates of a vector before the reflection and (w_1, w_2) are coordinates of the vector after the reflection;
- $(v_1, v_2) \in l_1, (w_1, w_2) \in l_2$, and line l_1 is billiard reflection of line l_2;

- $v_1^2 + v_2^2 = w_1^2 + w_2^2$.

The reflection in topological billiard is defined in the same way for common boundary arcs. If the mass point on one of the sheets of the billiard hits the gluing edge, it reflects and continues motion along the second sheet.

Above-described reflection preserves two integrals: the caustic parameter Λ and the vector of Euclidean velocity v_E. It turns out that this motion is integrable.

Fixing v_E we obtain an isoenergy surface Q^3 - 3-dimensional manifold in 4-dimensional phase space. Changing the value of Λ, we shall obtain the foliation of Q^3. The proof of the fact, that common layers in this foliation are tori, is obtainable by geometric methods.

Figure 1 shows various Fomenko-Zieschang invariants for different topological billiards.

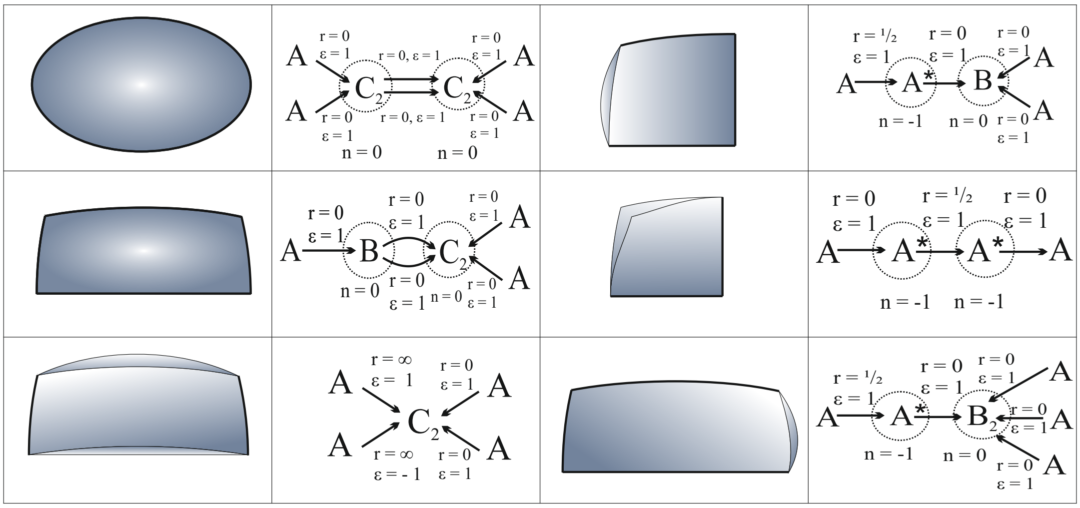

Fig. 1. Fomenko-Zieschang invariants for different topological billiards, obtained as a result of gluing two similar exemplars of simple billiards: billiards in left column and corresponding invariants in right column

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NORMALISATION OF THE PARABOLIC EQUATION WITH SMALL DIFFUSION AND STRONG NONLINEARITY

Kashchenko I. S.

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
iliyask@uniyar.ac.ru

Consider a system of nonlinear equations of parabolic type

$$\frac{\partial u}{\partial t} = \varepsilon D \frac{\partial^2 u}{\partial x^2} + (A_0 + \mu A_1)\, u + F\left(u, \frac{\partial u}{\partial x}\right), \tag{1}$$

$$u(t, x + 2\pi) \equiv u(t, x). \tag{2}$$

Here $u \in \mathbb{R}^n$, all the eigenvalues of the matrix D have positive real parts, the nonlinear vector-valued function $F(u, v)$ is linear in each argument, and ε and μ are small positive parameters: $0 < \varepsilon, \mu \ll 1$.

Let us consider the problem of investigating the local (in the neighborhood of the zero equilibrium state) the dynamics of the boundary value problem (1), (2), ie, the behavior as $t \to \infty$ of all solutions (1), (2) with initial conditions from a neighborhood of zero that is sufficiently small (but does not depend on small parameters).

In the case when the nonlinearity depends only on u (does not depend on $\frac{\partial u}{\partial x}$), the corresponding results are given in [1].

An important role in the problems of local analysis is played by the location of the eigenvalues of the family of matrices $A(z) = A_0 - zD$ that depend on the parameter $z \in [0, \infty)$. In the case when for all considered

z all eigenvalues of $A(z)$ have negative real parts, all solutions from some fixed neighborhood of zero tend to zero as $t \to \infty$. If there is a $z_0 \geq 0$ such that the matrix $A(z_0)$ has an eigenvalue with positive real part, then the posed problem becomes nonlocal. Here we will study the question of local dynamics (1), (2) in the case when $A(z)$ has no eigenvalues in the right complex half-plane for all $z \in [0, \infty)$, and there is a $z_0 \geq 0$ such that $A(z_0)$ has an eigenvalue with zero real part. We note at once that in this case the characteristic equation

$$\det(A_0 + \mu A_1 - \varepsilon k^2 D - \lambda I) = 0 \quad (k \in \mathbb{Z}),$$

has infinitely many roots whose real parts lie in some neighborhood of zero, whose length tends to zero as $\varepsilon, \mu \to 0$. So, we can say that the this critical case has infinite dimension.

It will be shown that the local dynamics (1), (2) strongly depends on the relations between the small parameters ε and μ. In this connection, it is convenient to assume that for some constants $c > 0$ and $\alpha \in (0, 1]$ we have

$$\mu = c\varepsilon^{\alpha}.$$

Note that the results differ fundamentally from those given in [1], For example, in the case when the family of matrices $A(z)$ has a pair of purely imaginary eigenvalues, the local dynamics (1), (2) can be described by the following quasinormal form [2]

$$\frac{\partial \xi}{\partial \tau} = \varkappa^2 d \frac{\partial^2 \xi}{\partial y^2} + c(A_1 a, b)\xi + \varkappa^2 \left[\sigma_1 \xi \left| \frac{\partial \xi}{\partial y} \right|^2 + \sigma_2 |\xi|^2 \frac{\partial^2 \xi}{\partial y^2} + \sigma_3 \xi^2 \frac{\partial^2 \overline{\xi}}{\partial y^2} + \sigma_4 \overline{\xi} \left(\frac{\partial \xi}{\partial y} \right)^2 \right], \tag{3}$$

$$\xi(\tau, y + 2\pi) \equiv \xi(\tau, y). \tag{4}$$

Theorem *Let for some $\varkappa$ problem (3), (4) has periodic by τ solution $\xi_0(\tau, y)$. Then boundary problem (1), (2) has a solution $u(t, x, \varepsilon)$ asymptotic with respect to the discrepancy up to $O(\varepsilon^{(1+\alpha)/2})$ uniformly with respect to $t \geq 0$, $x \in [0, 2\pi]$, for which*

$$u(t, x, \varepsilon) = \varepsilon^{\frac{1}{2}} [\xi_0(\varepsilon^{\alpha} t, (\varkappa \varepsilon^{\frac{\alpha-1}{2}} + \theta)x) \exp{(i\omega t)} a +$$

$$+ \overline{\xi_0}(\varepsilon^{\alpha} t, (\varkappa \varepsilon^{\frac{\alpha-1}{2}} + \theta)x) \exp{(-i\omega t)} \overline{a}].$$

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FROM THE CHARACTER TABLE OF A GROUP TO THE STRUCTURE OF A GROUP

Kazarin L. S.

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
lsk46@mail.ru

Let $G = \{g_1, g_2, \dots, g_n\}$ be a finite group of order n. Let $\{\chi_1, \chi_2, \dots, \chi_k\}$ be its irreducible ordinary characters and $T = (\chi_i(g_j))$ be its character table of G. In this talk we discuss which properties of G can be retrieved from the table T. Clearly, we may determine immediately the class number k of G and its order $n = \sum_{i=1}^{k} \chi_i(1)^2$, provided g_1 is the unite element of G. After the determining the canonical decomposition of $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, where $p_i (1 \le i \le s)$ are all prime divisors of n and α_i – natural numbers, we have determined the order $|G|_p$ of a Sylow p-subgroups of G for every prime p dividing $n = |G|$. Also we may introduce the prime graph $\Gamma(G)$, known also as a Gruenberg-Kegel graph, with vertices $V(G) = \{p_1, p_2, \dots, p_s\}$ and edges $(p_i, p_j) \in E(G)$. We say that the vertices p_i and p_j are adjacent if G has an element x whose order is divisible by $p_i p_j$.

The main problem here is the distribution of the representatives of the conjugacy classes of G by their orders. For each element $x \in G$ it is known the how to compute the order of its centralizer $C_G(x)$. For this we use the orthogonality relation: $|C_G(x)| = \sum_{i=1}^{k} |\chi(x)|^2$.

Usually this gives not big choice for the possibilities for the order of x. In fact, often it is interesting to determine for every prime p dividing $|G|$ the sets Σ_p of elements whose orders are coprime to p and the set of p-elements. One of the possibilities is as follows. Starting from an arbitrary element $g \in G$ we will decide, wether g is a p-element, Σ_p-element, or a mixed (i.e. a nontrivial product of an element in Σ_p and a p-element. It is well-known that the value of the character χ on $g \in G$ of order m is the sum of the m-th roots of unity. If $m = p^a t$, where

$(p, t) = 1$, then $\chi(g)^{p^a} \equiv \chi(g^{p^a}) \pmod{\mathfrak{p}}$ for some maximal ideal in the ring of all algebraic integers, containing a prime p. It follows that if g is a p-element, then $\chi(g)^{|G|_p} \equiv \chi(1) \pmod{p}$ for every irreducible character χ of G. This gives the possibility to separate p-elements from others. Using this process several times, we found at the end for every p dividing $|G|$ the set of p-elements and the set Σ_p of elements whose orders are coprime with p (p-regular elements).

Now there is a possibility to reconstruct the prime graph $\Gamma(G)$. For every p-element x of G we compute the order $|C_G(x)|$ and find all prime numbers $\pi(C_G(x))$ dividing $|C_G(x)|$. The union of these primes gives us the adjacent vertices for the prime p. At the end we have a complete information concerning the prime graph $\Gamma(G)$.

Another important information can be used from the set $\omega(G)$ of the orders of elements in the group G. The activity in this field during last 40 years gives a lot of results that gives a possibility to recognize a finite simple group G by this characteristic.

Clearly, it is possible to recognize that the group with the character table T is simple. This is determined by the property that there exists only one nontrivial normal subgroup $G' = G$. In general, the character χ determined its kernel $K = \{g \in G | \chi(g) = \chi(1)\}$. Thus for the simple nonabelian group there is just one such character, the principal character of G. The number of characters of degree 1 is the index of a commutator subgroup G' of G.

Assume that N is a normal subgroup of G, which corresponds to an irreducible character χ. This character has a kernel N consisting of all elements $g \in G$ such that $\chi(g) = \chi(1)$. In fact, we have only the representatives of the conjugacy classes of G, that are contained in N. Since we know the sizes of these classes ($|G : C_G(g_i)|$), we can compute $|N|$ and, therefore, the order of the group G/N. Afterwards, this gives the possibility to construct the character table of G/N and repeat the steps to understand the structure of the corresponding factor-group.

Of course, there are many further important properties and characteristic of a group, that can be retrieved from its character table T. First of all, the structure constants a_{ijr} that determined the products of the class sums C_i and C_j of all elements conjugate with g_i and g_j respectively. Also, the tensor product of the representations affording the irreducible characters χ_i and χ_j leads to the other structural constants

c^r_{ij} such that

$$\chi_i\chi_j = \sum_{r=1}^{k} c^r_{ij}\chi_r.$$

The case, when the constants c^r_{ij} for the group G (whose all elements are real) belongs to the set $\{0, 1\}$ was described first by E.P.Wigner in 1940 and the corresponding group was called an SR-group. It was proved by the author and his students that the finite SR-group is soluble.

Recently some authors has found the information concerning the existence of a nilpotent Hall subgroup of a finite group.

Remark. It should be noted that non-isomorphic groups may have the same character tables: the classical examples - groups quaternion and the dihedral group of order 8.

The aim of the authors is to describe some other results that gives the usage of the character table of a group for understanding the structure of a finite group.

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REALIZATION OF 3-ATOMS BY INTEGRABLE BILLIARD’S BOOKS

Kharcheva I. S.

Lomonosov Moscow State University, Moscow, Russia;
irina.harcheva1@yandex.ru

Let us consider a standard billiard problem in some fixed domain. A particle moves in a straight line within the domain. When the particle hits the boundary it reflects from it without velocity loss. This dynamic system contains the first integral – the scalar square of the velocity vector. In some special cases such a system has another integral.

One of these cases is a billiard in the domain bounded by confocal quadrics. The second special integral can be described by the following feature of trajectories: the straight lines containing the segments of the polygonal billiard trajectory are tangents to a certain quadric (ellipse or hyperbola).

This dynamical system has 4-dimensional phase space and two integrals. One of them is a Hamiltonian. Integrable Hamiltonian systems with 2 degrees of freedom have Fomenko-Zieschang invariants [1]. Such invariants allow us to speak about the equivalence between closures of trajectories – Liouville equivalence. Up to Liouville equivalence, such systems have been studied in details in [2] by V. Dragovich, M. Radnovich, and in [3, 4] by V.V. Fokicheva.

A billiard’s book is a generalization of these billiards. Such type of billiards formed by gluing a few classical billiard domains along pieces of their boundaries. The special case where we glue two domains called a topological billiard and was researched by V.V. Fokicheva [5].

Researching billiard’s books we try model famous integrable systems in terms of Fomenko-Zieschang invariants. The Fomenko conjecture about modeling Fomenko-Zieschang invariants using billiard’s books and new results that confirm the part of the conjecture will be presented.

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TOPOLOGY OF THREE DIMENSIONAL INVARIANT MANIFOLDS AND KOVALEVSKAYA INTEGRABLE CASE

Kibkalo V. A.

Lomonosov Moscow State University, Moscow, Russia;
slava.kibkalo@gmail.com

I.V. Komarov in his paper [1] showed that the Kovalevskaya integrable case in rigid body dynamics can be included in a one parameter family of integrable Hamiltonian systems on the pencil of Lie algebras so(3, 1) − e(3) − so(4) with real parameter $æ \in \mathbf{R}$. The Kovalevskaya top was realized as a system on e(3) for $æ = 0$. The following bracket on $\mathbf{R}^6$ depends on æ:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = æ\varepsilon_{ijk} J_k, \qquad (1)$$

where ε_{ijk} is the sign of permutation $\{123\} \to \{ijk\}$, $æ \in \mathbf{R}$. When $æ > 0, æ = 0, æ < 0$ this bracket coincides with the Lie–Poisson brackets for the Lie algebras so(4), e(3), so(3, 1) respectively. These brackets have common Casimir functions

$$f_1 = (x_1^2 + x_2^2 + x_3^2) + æ(J_1^2 + J_2^2 + J_3^2), \quad f_2 = x_1 J_1 + x_2 J_2 + x_3 J_3. \quad (2)$$

The Hamiltonian and an additional integral of these systems are equal to

$$H = J_1^2 + J_2^2 + 2J_3^2 + 2c_1 x_1, \qquad (3)$$

$$K = (J_1^2 - J_2^2 - 2c_1 x_1 + æc_1^2)^2 + (2J_1 J_2 - 2c_1 x_2)^2. \qquad (4)$$

We consider three-dimensional submanifolds which are common level surfaces of first integrals of the system. Topology type of them (class of diffeomorphic manifolds) can be determined by several ways. One of them is analysis of Fomenko-Zieschang invariants. They were calculated in [2] for the classical Kovalevskaya case and by speaker for the case of the Lie algebra so(4).

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BIFURCATIONS OF SPATIALLY INHOMOGENEOUS AND PERIODIC SOLUTIONS FOR NONLOCAL EROSION EQUATION

Kovaleva A. M.

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
anastasia2kovaleva@gmail.com

We consider a periodic boundary-value problem for a nonlinear equation with the deviating spatial argument. This equation is called a spatially nonlocal erosion equation. It describes the formation of undulating surface relief under the influence of ion bombardment and can be interpreted as a development of the well-known Bradley-Harper model, which is based on Sigmund's theory of sputtering [1].

Consider the equation

$$u_t = au_{xx} - cw_x + b(u-w) + b_1(u-w)w_x + b_2(w_x)^2 + b_3(u-w)(w_x)^2 + b_4 w_x^3,$$

where $u = u(t,x)$ describe the height of a substrate surface at time t, depending on a one-dimensional space variable x, $w = u(t, x - h), h \in \mathbb{R}, h > 0$. The coefficients of equation depend on parameters of the problem, such as the angle of incidence of the ion beam on the unperturbed substrate surface and the substrate material.

We will consider the nonlocal erosion equation with conditions

$$u(t, x+2\pi) = u(t,x),\ u(0,x) = f(x),\ f(x) \in H_2^2.$$

This problem was studied in case when $0 < h \ll 1$.

In order to solve the occurring bifurcation problems there were used the investigation methods of dynamical systems with an infinite-dimensional phase space (a space of initial conditions) such as: the method of integral manifolds, the method of Poincare-Dulac normal forms and asymptotic methods of analysis. In particular, asymptotic formulas were obtained for solutions which describe nonhomogeneous undulating surface relief. The question about the stability of these solutions was studied. And the analysis of normal form was given [2].

It is shown that the nonhomogeneous surface relief can occur when the stability of the homogeneous states of equilibrium changes. In this boundary value problem the loss of stability can occur at the higher modes and a number of such modes. In the case when the deviation h

is small enough, then surface relief with small wavelength can appear. This relief can actually be interpreted as nanorelief. The latter is quite important for applications in modern nanoelectronics.

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INTEGRABLE SYSTEMS AND LOW-DIMENSIONAL AFFINE MANIFOLDS

Kozlov I. K.

MSU, Moscow, Russia; ikozlov90@gmail.com

It is well-known, since at least the paper of J. J. Duistermaat [1], that integrable Hamiltonian systems define a structure of an affine manifold on the base of the fibration in a neighbourhough of a regular point. More precisely, these structures are of the so-called integral affine manifolds, i.e. for a proper atlas all the transition maps are affine maps $x \to Ax+b$ with an integer matrix A (i.e. all components of A belong to $\mathbb{Z}$).

In the talk we will discuss some properties of low-dimensional integral affine manifolds. In particular we will discuss the classification of two-dimensional integrable affine manifolds that can be found in [2, 3] including the following fact from [2]:

Theorem 1. *Among compact two-dimensional surfaces only the torus and the Klein bottle admit an integrable affine structure.*

After that we will discuss the analogous results for the three-dimensional integrable affine manifolds.

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INTEGRABLE SYSTEMS AND TENSOR INVARIANTS

Kozlov V. V.

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia vvkozlov@mi.ras.ru

Tensor invariants are tensor fields on the phase space invariant with respect to the flow generated by the vector field of the system. Simple examples include first integrals, fields of symmetries, volume forms of the invariant measures etc. The vector field of an autonomous system of differential equations is always a (trivial) tensor invariant.

The general idea is that for the integrability of a system of differential equations in an n-dimensional phase space we need $n-1$ independent non-trivial tensor invariants (there are valence-related dependencies between them). All known fundamental results on the integrability of differential equations (the Euler-Jacobi theorem on the integrating factor, the Lie theorem on the solvable algebra of symmetries, the Liouville theorem on the integrability of Hamiltonian systems etc.) satisfy this scheme.

The connection of this approach with the asymptotic method of Kovalevskaya-Lyapunov-Painlevé will be discussed. A new Euler-Jacobi-Lie theorem on the integrability, which embraces fields of symmetries and invariant forms, is also presented.

MULTISTABILITY AND CHAOTIC OSTILLATIONS IN A DELAYED NICOLLSON BLOWFLIES EQUATION

Kubyshkin E. P.[1], Moriakova A R.[2]

[1]*P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;* kubysh.e@yandex.ru

[2]*P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;* alyona_moryakova@mail.ru

We consider a differential-difference equation of the form

$$\dot{N}(t) = -\gamma N(t) + pN(t-\tau)e^{-aN(t-\tau)}, \quad t \geq 0, \tag{1}$$

proposed in [1] to describe the population dynamics of Nicholson's blowflies. Here $N(t)$ is the size of the population at time t, p is the maximum per capita daily egg production rate, $1/a$ is the size at which the population reproduces at the maxi- mum rate, γ is the per capita daily adult death rate, and τ is the generation time.

The equation (1) for $a > 0$ и $p > \gamma$ has single positive equilibrium state $N^* = a^{-1}\ln(p/\gamma)$.

Let us move to dimensionless variables in equation (1), setting $N(t) = N^* + a^{-1}x(t), (\gamma\tau)^{-1} = \varepsilon_1, aN^* - 1 = b$, and normalizing $t \to t\tau$. As a result, we obtain an equation in dimensionless variables

$$\varepsilon_1\dot{x}(t) + x(t) + bx(t-1) + f(x(t-1)) = 0, \qquad (2)$$

$$f(x) = aN^*(1 - x - e^{-x}) + x(1 - e^{-x}).$$

We study periodic solutions that bifurcate from N^* and their evolution with the changing of the patameters ε_1 and b with the assumption $\varepsilon_1 \ll 1$. It is shown that from the equilibrium state N^* a certain number (depending on the nature of the parameter variation) of stable (and unstable) periodic solutions can be bifurcated simultaneously. There is a bifurcation of multistability.

With the further changing of the bifurcation parameters, each periodic solution through a series of the period doubling bifurcations becomes a chaotic attractor. In the dynamics of the solutions of the equation chaotic multistability is observed.The obtained results make it possible to understand the structure of the phase space of the equation and explain the nature of the oscillations observed in the experimental data of Nicholson. The general scheme for solving such problem for equations of the form (2) is presented in [2].

Bifurcations of periodic solutions are related to the loss of stability of the equilibrium state N^*, which is determined by the location of the roots of the characteristic equation

$$\varepsilon_1\lambda + 1 + b\exp(-\lambda) = 0, \qquad (3).$$

For $b < 1$ and $0 < \varepsilon_1 < \varepsilon_0$ all roots of the equation(3) lie in the left complex half-plane, for $b > 1$ there are roots, lying in the right complex half-plane. Border is the point $b = 1$. Let $b = 1 + \varepsilon_2,\ |\varepsilon_2| \ll 1$.

Theorem 1. *There is $\varepsilon_0 > 0$, that for $|\varepsilon| \leq \varepsilon_0$ ($\varepsilon = (\varepsilon_1, \varepsilon_2),\ |\varepsilon| = (\varepsilon_1^2 + \varepsilon_2^2)^{1/2}$) the entire set of roots of equation (3) is given by*

$$\lambda_k(\varepsilon) = i\pi k + \ln(1 + \varepsilon_2) + \lambda^1(i\pi k + \ln(1 + \varepsilon_2); \varepsilon_1),$$

$k = \pm 1, \pm 3, \dots,$ $\lambda_{-k}(\varepsilon) = \bar{\lambda}_k(\varepsilon)$, *where* $\lambda^1(w;\varepsilon) \equiv -\ln(1+\varepsilon_1(w - \ln(1+\varepsilon_1(w - \ln(1+\varepsilon_1(w - \dots))))))$ $(\ln w = \ln|w| + i \arg w,\ -\pi < \arg w < \pi)$ *analytical function for* ε *and* w *for* $0 \le |\varepsilon| < \varepsilon_0,\ |Rew| < \delta_0$.

We introduce the spaces l_2 и l_2^1 sequences of the form $(z_1, z_{-1}, z_3, z_{-3}, \dots),$ $z_n \in \mathbb{C},$ $z_{-k} = \bar{z}_k$, satisfying conditions $||z||^2_{l_2} = \sum_{k=1,3,\dots} |z_k|^2 < \infty,$ $||z||^2_{l_2^1} = \sum_{k=1,3,\dots} |\lambda_k(\varepsilon) z_k|^2 < \infty$ $(|z|^2 = z\bar{z})$. Denote $S(r_0) = \{z \in l_2, ||z||_{l_2} < r_0\}$,$S^1(r_0) = \{z \in l_2^1, ||z||_{l_2^1} < r_0\}$.

In l_2 we consider the system of differential equations

$$\dot{z}_k = \lambda_k(\varepsilon) z_k + Z_k^{(3)}(z;\varepsilon) + Z_k^*(z;\varepsilon) \equiv Z_k(z;\varepsilon),\ k = \pm 1, \pm 3, \dots, \quad (3)$$

where the nonlinear operator $Z(\cdot) = (Z_1(\cdot), Z_{-1}(\cdot), Z_3(\cdot), Z_{-3}(\cdot), \dots)$ $(Z_{-k}(\cdot) = \bar{Z}_k(\cdot)) : S^1(r_0) \to S(r_0)$ is analytical for ε and z for $0 < \varepsilon < \varepsilon_0,$ $z \in S^1(r_0)$. Wherein $Z^{(3)}(\cdot) : S^1(r_0) \to S^1(r_0)$ is homogeneous form of order 3; $||Z^*(\cdot)||_{l_2} = O(||z||^5_{l_2^1})$.

Denote $S(R_0) = \{u(s) \in \mathbb{C}[-1, 0],\ ||u(s)||_C < R_0\}$.

Theorem 2. *There are* $\varepsilon_0,$ $R_0,$ $r_0 > 0$ *and the system of differential equations of the form (3), that for* $0 \le |\varepsilon| < \varepsilon_0$ *the behavior of solutions of equation (2) with initial conditions from* $S(R_0)$ *for* $t \to \infty$ *is completely determined by the behavior of the solutions of the system of equations (3) with initial conditions from* $S^1(r_0)$*. The system of equations (3) is determined uniquely.*

An algorithm for constructing a system of differential equations (3) is proposed.

We introduce real sequences of the form $\rho = (\rho_1, \rho_3, \dots),$ $\rho_k \ge 0,$ $k = 1, 3, \dots,$ $\sum_{k=1,3,\dots} k^2 \rho_k^2 < \infty$ и $\theta = (\theta_1,\ \theta_3, \dots),$ $0 \le \theta_j < 2\pi,$ $j = 1, 3, \dots$ Represent ε_1 and ε_2 in the form $\varepsilon_1 = \zeta \cos\psi,$ $\varepsilon_2 = \zeta^2 \sin^2 \psi sign\psi,$ $\zeta = (\varepsilon_1^2 + \varepsilon_2 sign \varepsilon_2)^{1/2},$ $-\pi/2 \le \psi \le \pi/2$.

The structure of the system of equations (3) allows us to introduce in place of z_k $(k = \pm 1, \pm 3, \dots)$ one "fast" variable and countable number of "slow" variables of the form ρ and θ. Averaging the resulting system of equations for the "fast" variable, we obtain a system of equations whose limiting part (for $\zeta \to 0$) will be look like

$$\dot{\rho}_k = \gamma_k(\psi)\rho_k + R_k(\rho,\ \theta)(\gamma_k(\psi) = \sin^2 \psi sign\psi - \cos^2 \psi (\pi k)^2/2),$$

$$\dot{\theta}_k = \Theta_k(\rho, \theta),\ k = 1, 3, \dots, \quad (4)$$

where functional $R_k(\cdot), \Theta_k(\cdot)$ 2π is periodical for θ_j, $R_k(\cdot)$ is a homogeneous form of order 3 in ρ_j.

Theorem 3. *To each exponentially stable (unstable) equilibrium state $\rho^*(\psi), \theta^*(\psi)$ of the system of equations (4) for $0 < \zeta < \zeta_0$ corresponds a periodic solution of equation (2) of the same kind of stability. This periodic solution is analytic function of the parameter ζ, the main part of which is determined by the indicated equilibrium state.*

Theorem 3 allows us to reduce the problem of constructing of periodic solutions of equation (2) bifurcating from the equilibrium state N^* to the problem of finding the equilibrium states of the system of equations (4). The states of equilibrium make it possible to construct asymptotic formulas for periodic solutions that determine the initial conditions for the numerical obtaining of exact solutions of the equation and the study of the further evolution of periodic solutions when the parameters of the equation are changing.

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CLASSIFYING SUPERINTEGRABLE BERTRAND MECHANICAL SYSTEMS

Kudryavtseva E. A.

Lomonosov Moscow State University, Moscow, Russia;
eakudr@mech.math.msu.su

The problem of description of superintegrable systems (i.e., systems with closed trajectories in a certain domain) in the class of spherically symmetric natural mechanical systems goes back to Bertrand and Darboux. Dynamical and geometric properties of such systems have been studied by many mathematicians (Killing, Besse, Perlick, Kozlov, Borisov, Mamaev, Santoprete, Ballesteros, Enciso, Herranz, Ragnisco). However in full generality, the problem of description of all such systems remained open because of the so-called "problem of equators" (here, by an equator we mean a parallel that is a geodesic).

Let us proceed with precise statements.

Bertrand proved [1] that, in Newtonian mechanics, the Kepler-Coulomb potential $V_I(r) = -a/r+b$ and the harmonic oscillator (Hooke's) potential $V_{II}(r) = ar^2 + b$ ($a, r \in \mathbb{R}$, $a > 0$) are distinguished by the property that:

- all the bounded non-singular trajectories are closed and
- there exist non-circular non-singular closed orbits.

Natural mechanical systems possessing the above property will be called *Bertrand systems*. Other definitions of Bertrand-type systems, as well as their classifications, are discussed in [2]—[4].

Darboux [5] and Perlick [6] independently extended Bertrand's theorem as follows. They obtained a complete description of all spherically symmetric Bertrand systems, whose underlying Riemannian manifolds of revolution *have no equators*. Similarly to the Bertrand [1] result (the case of a punctured Euclidean plane), the systems from their classification form two finite-parametric families:

(I') the Darboux-Perlick family of systems with the Kepler-Coulomb potentials and

(II') the Darboux-Perlick family of systems with the harmonic oscillator (Hooke's) potentials.

Clearly, the Darboux-Perlick families (I') and (II') do not contain such famous Bertrand systems on surfaces of revolution *having equators* as, e.g.,

(I) the systems with the Kepler-Coulomb potential on the round spheres (or their “rational branched coverings”),

(II) the systems with the harmonic oscillator (Hooke's) potential on pear-shaped surfaces of revolution whose equators split them into two subsurfaces from the Darboux-Perlick family (II'), and

(III) the systems with zero potential on (pear-shaped or spherical) Tannery surfaces of revolution classified by Besse [7].

We give a complete description of all spherically-symmetric Bertrand systems [8]. In particualar, we do not assume that the underlying Riemannian manifold of revolution has no equators. The systems from our list are parametrized by smooth functions, so they form an infinite-dimensional set. We prove, in particular, the following

Theorem. *For any spherically-symmetric Bertrand system, its restriction to the cotangent bundle of the union of all non-circular non-singular closed orbits coincides with the restriction to a "belt" of one of the systems from the families (I'), (II'), (I), (II), (III) from above.*

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VIV MODELLING FOR FLEXIBLE STRUCTURES USING WAKE OSCILLATOR METHOD

Kurushina V.[1,2], Pavlovskaia E.[1], Wiercigroch M.[1]

[1]*Centre for Applied Dynamics Research, School of Engineering, University of Aberdeen, Kings College, Aberdeen, Scotland, United Kingdom, AB24 3FX;*

[2]*Department of Transport of Hydrocarbon Resources, Institute of Transport, Industrial University of Tyumen, 38 Volodarskogo Street, Tyumen, Russia, 625040;* v.kurushina@outlook.com

Structures like risers, free pipeline spans and umbilicals in the water, or cables, tall buildings, suspension bridges in the air, can be subjected to an additional loading due to vortex-induced vibration (VIV). Influence of the fluctuating lift and drag forces should be taken into account at the design stage of these slender structures. Currently, the models involved in VIV prediction still lack precision or require significant computational time.

The current study is focused on improving VIV prediction accuracy. The wake oscillator method, used in this research, provides advantageous computational time by using self-excited limit cycle oscillators to model force fluctuations. However, the precision of modelling depends on the choice of empirical constants which allow simplifying the calculations.

This research includes developing a new model of wake oscillator type for the case of a flexible structure moving in both directions of the flow and across the flow. The model is based on the results of studies [1–3]. The scope of research is limited to a uniform flow velocity profile as a necessary step before investigating VIV in sheared flow in the future. The authors consider 6 various nonlinear damping types for the wake oscillator equations and conduct calibration of the empirical coefficients using the experimental data [4]. Application of alternative damping terms results not only in improving prediction accuracy but also in qualitative differences in generated trajectories of motion and modulated displacement signals. As the outcome of this study, the model versions with Van der Pol dampings are the most suitable for VIV predictions for flexible structures.

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BIFURCATIONS IN 3 D.O.F. INTEGRABLE HAMILTONIAN SYSTEMS

Lerman L. M.

Lobachevsky State University of Nizhny Novgorod, Russia;
lermanl@mm.unn.ru

Studying integrable Hamiltonian systems always leads to an investigation of bifurcations. This is related with the fact that in such systems for the related Poisson action all its singular orbits (of dimension lesser than the half of the manifold dimension) are met in families. For instance, periodic orbits being 1-dimensional orbits of the induced Poisson action belong to a 1-parameter families, 2-dimensional Lagrangian tori belong to 2-parameter families, etc. This implies that if one moves along the family an orbit being more degenerate (in transverse direction) than neighboring orbits can be met and hence one may expect branching the family. Also bifurcations are met at the study of families of integrable systems, then parameters of the family play a similar role. It is important to stress that common tool to study integrable systems uses some

assumptions on the linearized system at the related Poisson orbits (like to be a Cartan algebra for the related set of commuting integrals, etc). Such properties are usually violated at the bifurcation and one needs to use another tool to study the related orbit structure.

We discuss this topic for 3 degree of freedom integrable Hamiltonian systems. In this case (if no outer parameter exist) the related integrable system can contain 1-parameter families of periodic orbits and 2-parameter families of Lagrangian 2-tori. Thus one can meet degenerations of codimension 1 and two. Related structures of Liouville foliations in saturated neighborhoods of the degenerated orbits and their "perestroika's" will be presented.

THE AUTOWAVE FRONT STATING IN THE DISCONTINUOUS MEDIA

Levashova N. T.[1], **Orlov A. O.**[2]

[1]*Lomonosov Moscow State University, Faculty of Physics, Moscow, Russia;* natasha@npanalytica.ru

[2]*Lomonosov Moscow State University, Faculty of Physics, Moscow, Russia;* orlov.andrey@physics.msu.ru

We investigate a following problem

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2} - \varepsilon\frac{\partial u}{\partial t} = f(u,x), \quad -1 < x < 1,\ t > 0; \tag{1}$$

$$u(-1,t,\varepsilon) = \varphi_1,\ u(1,t,\varepsilon) = \varphi_3,\ t > 0; \quad u(x,0,\varepsilon) = u_{init}(x,\varepsilon),\ t > 0. \tag{2}$$

where ε is a small parameter, $f(u,x)$ is a function that undergoes discontinuity at point $x = -0.5$:

$$f(u,x) = \begin{cases} u - \varphi_1, & -1 \le x < -0.5; \\ (u-\varphi_1)(u-\varphi_2)(u-\varphi_3), & -0.5 \le x \le 1, \end{cases}$$

φ_i, $i = 1,2,3$ are constants for which the inequalities hold:

$$\varphi_1 < \varphi_2 < 0.5(\varphi_1 + \varphi_3) \tag{3}$$

and

$$u_{init} = \frac{\varphi_3(\varphi_2 - \varphi_1) + \varphi_1(\varphi_3 - \varphi_2)\exp\left(\dfrac{\varphi_3 - \varphi_1}{\varepsilon\sqrt{2}}(x - 0.3)\right)}{\varphi_2 - \varphi_1 + (\varphi_3 - \varphi_2)\exp\left(\dfrac{\varphi_3 - \varphi_1}{\varepsilon\sqrt{2}}(x - 0.3)\right)}.$$

We investigate the C^1 solution that has a front moving form and at a certain moment of time gets inside the area of attraction of the solution of the stationary problem [1]:

$$\varepsilon^2\frac{d^2 u}{dx^2} = f(u,x), \quad -1 < x < 1; \quad u(-1,\varepsilon) = \varphi_1,\ u(1,\varepsilon) = \varphi_3. \tag{4}$$

The inequality (2) guarantees that the front is moving in the direction opposite to the x-axis [2].

For the mentioned solution of problem (1) the asymptotic approximation $U_1(x,t,\varepsilon)$ of the first order in the exponents of ε is constructed. It is a sum of regular part and the transition layer functions:

$$U_1(x,t,\varepsilon) = \begin{cases} \varphi_1 + R_0(\xi_0,t) + \varepsilon R_1(\xi_0,t), & -1 \le x \le -0.5; \\ \varphi_1 + Q_0^{(-)}(\xi_*,t) + \varepsilon Q_1^{(-)}(\xi_*,t), & -0.5 \le x \le x_*; \\ \varphi_3 + Q_0^{(+)}(\xi_*,t) + \varepsilon Q_1^{(+)}(\xi_*,t), & x_* \le x \le 1. \end{cases} \quad (5)$$

Here we introduced the stretched variables $\xi_0 = (x+0.5)/\varepsilon$, $\xi_* = (x - x_*)/\varepsilon$, where x_* is the point of moving front localization.

To prove the existence of such a solution in the domain $(x,t) = [-1;1] \times \mathbb{R}^+$ we use the method of differential inequalities [3]. For that we use the method of upper and lower solutions. The upper solution, $\beta(x,t,\varepsilon)$ for problem (1) is the same as one for the stationary problem (3). The lower solution $\alpha(x,t,\varepsilon)$ is constructed as modification of the first order asymptotic approximation (4):

$$\alpha(x,t,\varepsilon) = \begin{cases} \varphi_1 + R_0(\xi_0,t) + \varepsilon R_1(\xi_0,t) - \varepsilon(r(\xi_0,t)+\mu), & -1 \le x \le -0.5; \\ \varphi_1 + Q_0^{(-)}(\xi_*,t) + \varepsilon Q_1^{(-)}(\xi_*,t) - \varepsilon(q^{(-)}(\xi_*,t)+\mu), & -0.5 \le x \le x_*; \\ \varphi_3 + Q_0^{(+)}(\xi_*,t) + \varepsilon Q_1^{(+)}(\xi_*,t) - \varepsilon(q^{(+)}(\xi_*,t)+\mu), & x_* \le x \le 1. \end{cases} \quad (6)$$

Positive constant μ and functions r, $q^{(\mp)}$ are chosen in a way that function (6) satisfies the definition of the lower solution given in [3].

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ACTION-ANGLE DUALITY VIA HAMILTONIAN REDUCTION

Marshall I.D

HSE, Moscow, Russia;

imarshall@hse.ru

I will give a survey type presentation describing how Hamiltonian reduction of trivially integrable free flows may give rise to so–called dual pairs of integrable systems. This will be followed by a description of results, obtained together with L. Feher, illustrating the construction.

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DEVELOPMENT OF THE DIFFERENTIAL INEQUALITIES METHOD FOR NONLINEAR ACTIVATOR-INHIBITOR SYSTEMS

Melnikova A. A.

Lomonosov Moscow State University, Moscow, Russia;

melnikova@physics.msu.ru

Models of the activator-inhibitor type are used in chemical kinetics, biophysics, and ecology to describe the interaction of components of different systems. The Murray and Meinhardt models [1, 2], which describe the formation of patterns in living nature, can be cited as an example. In the paper [3], a model based on the FitzHugh-Nagumo system is used to describe urboecosystems. These models are systems of nonlinear parabolic equations. In this paper, we consider a similar system in the case of small diffusion, which makes the problem singularly perturbed. Consider the initial-boundary value problem

$$\begin{cases} \varepsilon^4 u_{xx} - \varepsilon^2 u_t = f(u,v,x,\varepsilon), \quad \varepsilon^2 v_{xx} - \varepsilon^2 v_t = g(u,v,x,\varepsilon), \\ x \in (a,b), \quad t \in (0,T], \\ u_x(a,t) = u_x(b,t) = 0, \quad v_x(a,t) = v_x(b,t) = 0, \quad t \in (0,T], \\ u(x,0) = u_{init}(x), \quad v(x,0) = v_{init}(x), \quad x \in [a,b]. \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a small parameter, functions $f(u, v, x, \varepsilon)$ and $g(u, v, x, \varepsilon)$ are sufficiently smooth functions.

We consider the following hypotheses:

(H_1) The equation $f(u, v, x, 0) = 0$ treated as an equation for u has exactly three roots $u = \varphi^l(v, x)$, $u = \varphi^0(v, x)$, $u = \varphi^r(v, x)$ such that $\varphi^l(v, x) < \varphi^0(v, x) < \varphi^r(v, x)$ everywhere in the domain $(v, x) \in I_v \times [a, b]$, moreover $f_u(\varphi^{l,r}(v, x), v, x, 0) > 0$ and $f_u(\varphi^0(v, x), v, x, 0) < 0$. (Here I_v is some range of variable v.)

(H_2) Each of the equations $h^{l,r}(v, x) := g(\varphi^{l,r}(v, x), v, x, 0) = 0$, has a unique solution $v = v^{l,r}(x) \in I_v$, moreover, the inequalities $v^l(x) < v^r(x)$ and $h^{l,r}_v(v^{l,r}(x), x) > 0$ hold for all $x \in [a, b]$.

(H_3) The inequalities $f_v(u, v, x, 0) > 0$ and $g_u(u, v, x, 0) < 0$ hold everywhere in the domain $(u, v, x) \in I_u \times I_v \times [a, b]$. (Here I_u is some range of variable u.)

Condition 3 ensures the interaction of the components by activator-inhibitor type, where u-component is an activator and v-component is an inhibitor.

In this work, we prove the asymptotic approximation of the solution of the system (1) with an internal transition layer. The proof of the existence theorem under these conditions requires a substantial development of the method of differential inequalities in comparison with the works [4,5]. In [6], a close problem was considered for constructing upper and lower solutions for a model system of the FitzHugh-Nagumo type. In [7], for an example of the system (1) an asymptotics and numerical solution of the moving internal layer was obtained. The algorithm for constructing the asymptotic approximation for stationary and nonstationary problems coincides with that described in [4,5].

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INTEGRABLE BILLIARDS WITH NONCONVEX ANGLES ON THE BOUNDARY

Moskvin V. A.

Moscow State University, Moscow, Russia; aoshi.k68@gmail.com

A billiard is a dynamical system in which a particle alternates between motion in a straight line and specular reflection from a boundary. You can find overwiew of modern billiard researches in S.L. Tabachnikov’s book. [1] We used Fomenko’s theory [2] to describe topology of joint integrable surfaces. In this report I’m going to discuss billiards bounded by arcs of several confocal conics such that they contains nonconvex angles. Note that such billiards are integrable (the second integral is simply the confocal quadric parameter). Since there is no tangent in vertex of nonvconvex angle, the billiard reflection cannot be defined there in the usual way. The integral trajectory of the flow going into a singular point can not be extend for the all values of the parameter (time). In other words, the corresponding flow is not complete in the sense of the theory of differential equations. In contrast to the classical case of complete flows, the regular leaves of the Liouville foliation are the spheres with handles and punctures, rather than Liouville tori.

Theorem 1. *Any billiard’s domain bounded by arcs of several confocal conics such that they contain one nonconvex angle is equal to one of the 14 billiards represented on Fig. 1.*

Theorem 2. *Let S be the elementary billiard with nonconvex angles represented on the Fig.1. Then the rough molecules for this billiard has* *the form represented on the Fig.2. The edges with the marks correspond to the two-dimensional spheres with two handles without a point. Complexes* Γ_i^j*, where* $i \neq j$ *and* $i, j = 1; 2$ *are homeomorphic to a disjoint union of tori. To one of this tori the circle is glued along a point, whose projection onto the billiard domain goes to the vertex of a nonconvex angle.*

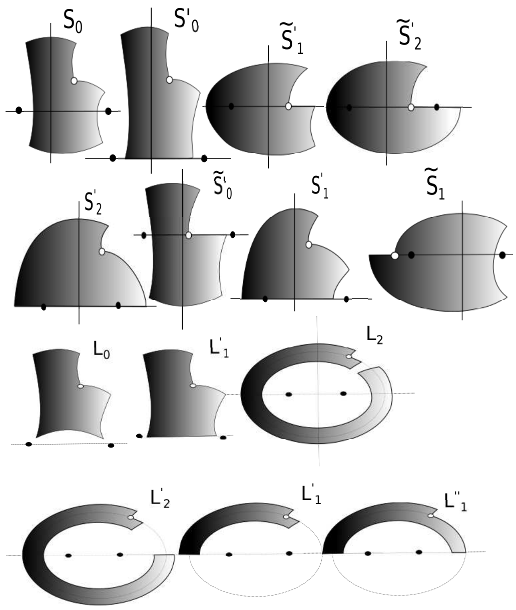

Fig. 1. Billiard's domains.

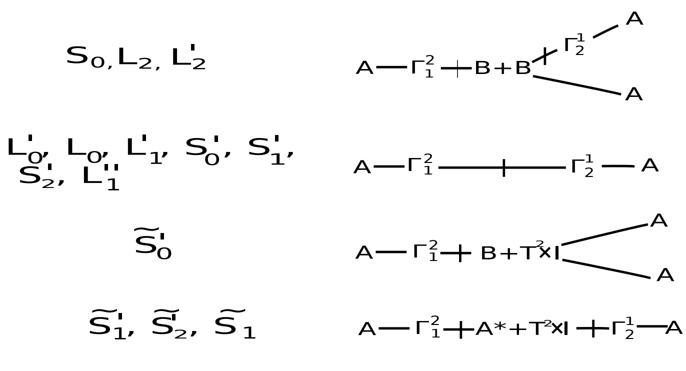

Fig. 2. Fomenko's molecules.

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PROPAGATION AND BLOWING-UP OF FRONTS IN REACTION-DIFFUSION-ADVECTION PROBLEMS WITH MODULAR ADVECTION

Nefedov N. N.

Department of Mathematics, Faculty of Physics, Lomonosov Moscow State University, Moscow, Russia;

nefedov@phys.msu.ru

We present recent results for some classes of IBVP (initial boundary value problem) for some classes of Burgers type equations where we investigate moving fronts by using the developed comparison technique. We also present our recent results of investigation of singularly perturbed reaction-advection-diffusion problems, which are based on a further development of the asymptotic comparison principle (see, [1-3]).

For these initial boundary value problems we proved the existence of fronts and give its asymptotic approximation. We illusrate our results by the problem

$$\varepsilon\frac{\partial^2 u}{\partial x^2} - A(u,x,t)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = f(u,x,t,\varepsilon), \quad x \in (0,1), 0 < t \le T,$$

$$u(0,t,\varepsilon) = u^0(t), \quad u(1,t,\varepsilon) = u^1(t), \quad t \in [0,T],$$
$$u(x,0,\varepsilon) = u_{init}(x,\varepsilon), \quad x \in [0,1].$$

An asymptotic approximation of solutions with a moving front is constructed in the case of modular and quadratic nonlinearity and nonlinear amplification [4]. The influence exerted by nonlinear amplification on front propagation and collapse is determined. The front localization and the collapse time are estimated.

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ON 0/1-MATRICES HAVING MAXIMUM DETERMINANT

Nevskii M. V.[1], **Ukhalov A. Yu.**[2]

[1]*P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
mnevsk55@yandex.ru

[2]*P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
alex-uhalov@yandex.ru

We give some new necessary conditions for maximality of 0/1-determinant of a given order n. Our approach lies in the field between linear algebra and geometry of convex bodies. At the same time, we demonstrate some examples how geometric results can be expressed by means of linear algebra. Basic ideas and relations are given in [2], [3], [4].

Let $n \in \mathbb{N}$, $Q_n = [0,1]^n$. By *a/b-matrix*, we mean a matrix, whose any element is equal to one of two numbers a or b. Values h_n and g_n are defined as maximum determinants of 0/1 and $-1/1$-matrices of order n respectively. Denote by ν_n the maximum volume of an n-dimensional simplex contained in Q_n. These numbers are connected by the equalities $g_{n+1} = 2^n\, h_n$, $h_n = n!\,\nu_n$, see [1].

If maximum 0/1-determinant of order n is known, it is possible to construct a maximum volume simplex in Q_n. Let us enlarge the row set of such a determinant by the row $(0,\ldots,0)$. Then simplex S with these vertices is contained in Q_n and has maximum possible volume. Nonzero vertices of an n-dimensional simplex in Q_n with maximum volume one can also obtain in the same way using the columns of a maximum 0/1-determinant of order n. On the other hand, if any n-dimensional simplex $S \subset Q_n$ with the zero-vertex has maximum possible volume among all simplices in Q_n, then the coordinates of its non-zero vertices, being written in rows or in columns, form maximum 0/1-determinants of order n.

Theorem. *Suppose* $\mathbf{M}$ *is an arbitrary nondegenerate* $0/1$*-matrix of order* n*. Let* $\mathbf{A}$ *be the matrix of order* $n+1$ *which appears from* $\mathbf{M}$ *after adding the* $(n+1)$*th row* $(0, 0, \ldots, 0, 1)$ *and the* $(n+1)$*th column consisting of* 1*'s. Denote* $\mathbf{A}^{-1} = (l_{ij})$*. Then the following propositions hold true.*

1. *For all* $i = 1, \ldots, n$,

$$\sum_{j=1}^{n+1} |l_{ij}| \geq 2. \tag{1}$$

2. *If* $|\det(\mathbf{M})| = h_n$*, then for all* $i = 1, \ldots, n$

$$\sum_{j=1}^{n+1} |l_{ij}| = 2. \tag{2}$$

3. *If for some* i *we have the strong inequality*

$$\sum_{j=1}^{n+1} |l_{ij}| > 2, \tag{3}$$

then $|\det(\mathbf{M})| < h_n$.

To sketch the proof, let us consider the simplex S with the node matrix coinciding with $\mathbf{A}$. Then the conditions above can be written in the terms of the so called axial diameters of S, see [4] for the detailes. Now it is sufficient to use the fact that all the axial diameters of a maximum volume simplex in Q_n are equal to 1.

It is easy to find out, that the inverse proposition to the second part of this statement is not true.

Note that 0/1-matrices having maximum determinant can be obtained from maximum determinant $-1/1$-matrices by the special procedure described in [5]. Therefore, the theorem also can be used to control the extremality of $-1/1$-matrices. As an example, we can show that biggest known by 2003 $-1/1$-determinant of order 101 is not maximum. This tremendous $-1/1$-matrix was constructed by William Orrick and Bruce Solomon, see http://www.indiana.edu/~maxdet/

d101.html. With the use of Wolfram Mathematica, we obtain that the corresponding matrices $\mathbf{M}$ and $\mathbf{A}$ satisfy the third condition of the theorem. Consequently, the huge value

$$|\det(\mathbf{M})| = |\det(\mathbf{A})| =$$

$$897408165779423029836382415865697614876239640580024570226669311523437 50$$

is yet strictly less than h_{100}. This also means that in this case the volume of simplex $S \subset Q_{100}$ is not maximum possible.

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HURWITZ NUMBERS AND MATRIX INTEGRALS LABELED WITH CHORD DIAGRAMS

Orlov A. Yu.

Institute of Oceanology, Moscow, Russia; orlovs@ocean.ru

We consider products of complex random matrices from independent complex Ginibre ensembles. The products include complex random matrices $Z_i, Z_i^\dagger$, $i = 1, \ldots, n$, and $2n$ sources (these are the complex matrices C_i, C_i^*, $i = 1, \ldots, n$, which play the role of parameters). We consider collections of products $X_1, \ldots, X_F$, constainted by the property, that each of the matrices of the set $\{Z_i C_i, Z_i^\dagger C_i^*,\ i = 1, \ldots, n\}$ is included only once on the product $X = X_1 \cdots X_F$. It can be represented graphically as a collection of v polygons with a total number of edges $2n$, and the polygon with number a encodes the order of the matrices in X_a. The matrices Z_i and $Z_i^\dagger$ are distributed along the edges of this collection of polygons, and the sources are distributed at their vertices. The calculation of the expected values involves pairing the matrices Z_i and $Z_i^\dagger$. There is a standard procedure for constructing a $2D$ surface by paiwise gluing edges of polygons, this procedure results to a graph embedded in the surface Σ_{E^*} of some Euler characteristic E^* (this graph also known as ribbon graph or fat graph). We propose a matrix model that generates spectral correlation functions for matrices X_a, $a = 1, \dots, F$ in the Ginibre ensembles, which we call the matrix integral, labeled network chord diagram. We show that the spectral correlation functions generate Hurwitz numbers H_{E^*} that enumerate nonequivalent branched coverings of Σ_{E^*}. The role of sources is the generation of ramification profiles in branch points which are assigned to the vertices of the embedded graph drawn on the base surface Σ_{E^*}. The role of coupling constants of our model is to generate ramification profiles in F additional branch points assigned to the faces of the embedded graph (the faces of the 'triangulated' Σ_{E^*}). The Hurwitz numbers for Klein surfaces can also be obtained by a small modification of the model. To do this, we pair any of the source matrices (in that case presenting a hole on Σ_{E^*}) with the tau function, which we call Mobius one. The presented matrix models generate Hurwitz numbers for any given Euler characteristic of the base surface E^* and for any given set of ramification profiles.

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ENUMERATION OF NON-DEGENERATE RANK 0 SINGULARITIES FOR INTEGRABLE HAMILTONIAN SYSTEMS WITH THREE DEGREES OF FREEDOM

Oshemkov A. A.

Lomonosov Moscow State University, Moscow, Russia;
a@oshemkov.ru

The problem of topological classification of singularities for integrable Hamiltonian systems has several aspects depending on the choice of a class of singularities and an appropriate equivalence relation. We consider non-degenerate singularities up to topological equivalence, i.e., up to a homeomorphism preserving leaves of the Liouville foliation defined by the integrals of the system.

By the Eliasson theorem, the local classification (not only topological but even symplectic) of non-degenerate rank 0 singularities for integrable Hamiltonian systems is described by their type, i.e., by a triple of numbers (n_1, n_2, n_3) which are equal respectively to the number of elliptic, hyperbolic, and focus components of a singularity. In the case of semi-local classification there is one more parameter k which is the complexity of a singularity, i.e., the number of singular points on the leaf.

The question on semi-local classification of singularities of various possible types was considered in many papers. Note that elliptic components can be ignored (from the topological point of view). The first non-trivial result was obtained by L.M. Lerman and Ya.L. Umanskii in 1984 (see [1]). They proved that there are exactly 4 saddle-saddle singularities (i.e., singularities of type (0,2,0) and rank 0) with one singular point on a singular fiber (i.e. of complexity 1). Saddle-saddle singilarities of complexity 2 were classified by A.V. Bolsinov and V.S. Matveev (see [2], [3]). It turns out that there are 39 topologically different singularities of such type. The complete answer in purely hyperbolic case, i.e., for singularities of type (0,n,0) and rank 0, was obtained by A.A. Oshemkov in paper [4], where a general algorithm of their enumeration was described and the lists of such singularities of small complexity were obtained. For example, there are 32 rank 0 singularities of type (0,3,0) and complexity 1. The classification of rank 0 singularities of type (0,0,m) for arbitrary complexity (i.e., purely focus singularities) was done by A.M. Izosimov [5].

The only type of singularities that is not investigated completely is the singularities with both hyperbolic and focus components. The simplest case when such situation is possible corresponds to saddle-focus singularities in systems with three degrees of freedom, i.e., singularities of type (0,1,1).

We will explain some approaches to semi-local classification of singularities of different types and give lists of saddle-focus singularities of small complexity.

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FORMAL AND EXACT DARBOUX TRANSFORMATIONS FOR VECTOR mKdV AND RELATED EQUATIONS

Papamikos G.

School of Mathematics, University of Leeds, Leeds, UK;
geopap1983@gmail.com

In this talk I will present some of the integrability properties of a multicomponent generalisation of the modified Korteweg de Vries equation. In particular, I will present a Lax structure of the equation which I will use to construct formal and closed form Darboux transformations. I will also present other integrability properties such as its hierarchy of conservation laws, generalised symmetries and its recursion operator. In the end I will remark on its solutions and its connection to a multicomponent version of the sine-Gordon equation and other related equations.

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HIGHER HIROTA DIFFERENCE EQUATIONS AND THEIR REDUCTIONS

Pogrebkov A. K.

Steklov Mathematical Institute RAS, Moscow, Russia; pogreb@mi.ras.ru
NRU "Higher School of Economics"

We derive and study higher analogs of the Hirota difference equations. We also consider integrable systems that appear as $1+1$ dimensional reductions of these difference equations and their continuous limits.

MULTIPLIERS OF AN ANTIPHASE SOLUTION IN A SYSTEM OF TWO COUPLED NONLINEAR RELAXATION OSCILLATORS

Preobrazhenskaia M. M.

Yaroslavl State University, Russia, Yaroslavl; rita.preo@gmail.com

Let's consider the system of non-linear differential-difference equations [1-3]

$$\begin{aligned} \dot u_1 &= [\lambda f(u_1(t-1)) + b\, g(u_2(t-h))\ln(u_*/u_1)]u_1, \\ \dot u_2 &= [\lambda f(u_2(t-1)) + b\, g(u_1(t-h))\ln(u_*/u_2)]u_2, \end{aligned} \tag{1}$$

which is used to model the association of two neurons with a synaptic connection. Here $h > 0$ characterizes a delay in the connection chain, $u_1(t), u_2(t) > 0$ are normalized membrane neurons potentials, the parameter $\lambda \gg 1$ characterizes an electrical processes rate in the system, $b = \text{const} > 0$, $u_* = \exp(c\lambda)$ is a threshold value for control interaction, $c = \text{const} \in R$, the terms $bg\big(u_{j-1}\big)\ln(u_*/u_j)u_j$ model synaptic interaction. The functions $f(u),\ g(u) \in C^2(R_+)$, $R_+ = \{u \in R : u \geq 0\}$, satisfy a following conditions:

$$f(0) = 1;\ f(u) + a, uf'(u), u^2 f''(u) = O\big(u^{-1}\big) \text{ as } u \to +\infty,$$
$$a = \text{const} > 0;\ \forall u > 0\ g(u) > 0,\ g(0) = 0;\ g(u) - 1,$$
$$ug'(u), u^2 g''(u) = O\big(u^{-1}\big) \text{ as } u \to +\infty.$$

For any natural n it is possible to choose the parameters a, b, c, h such that system (1) has a periodic antiphase solution containing n asymptotically high bursts on a period. The solution has a following form: $u_1^* = \exp(x^*(t)/\varepsilon)$, $u_2^* = \exp(x^*(t+\Delta)/\varepsilon)$, where $\Delta = T^*/2$, $\varepsilon = 1/\lambda \ll 1$, $x^*(t)$ is a T^*-periodic solution of equation

$$\dot{x} = f(\exp(x(t-1)/\varepsilon)) + b\,(c - x)g(\exp(x(t+\Delta-h)/\varepsilon)).$$

It is proved that the linearised on the antiphase solution system

$$\dot{\gamma}_1 = -bg(\exp(x_*(t+\Delta-h)/\varepsilon))\gamma_1 + A(x_*(t-1),\varepsilon)\gamma_1(t-1) + \\ + b(c - x_*(t))B(x_*(t+\Delta-h),\varepsilon)\gamma_2(t-h),$$

$$\dot{\gamma}_2 = -bg(\exp(x_*(t-h)/\varepsilon))\gamma_2 + A\big(x_*(t-\Delta-1),\varepsilon\big)\gamma_2(t-1) + \\ + b\big(c - x_*(t-\Delta)\big)B\big(x_*(t-h),\varepsilon\big)\gamma_1(t-h),$$

$$A(x,\varepsilon) = f'\big(\exp(x/\varepsilon)\big)\exp(x/\varepsilon)/\varepsilon,\ B(x,\varepsilon) = g'\big(\exp(x/\varepsilon)\big)\exp(x/\varepsilon)/\varepsilon,$$

has unit multiplier, two multipliers which modules are close to 1, but less than 1, and all other multipliers are exponentially small. It means that the antiphase solution (u_1^*, u_2^*) is exponentially orbitally stable.

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NEW FAMILY OF EXACT SOLUTIONS FOR THE RIEMANN EQUATION

Rassadin A. E.

Nizhny Novgorod Mathematical Society, Nizhny Novgorod, Russia;
brat_ras@list.ru

The Riemann equation

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = 0 \tag{1}$$

is known to be very popular model in theory of nonlinear waves [1].

On the other hand for equation (1) there is the following symmetry generator [2]:

$$\hat{X} = \xi(t,x,v)\frac{\partial}{\partial t} + [\zeta(v,x-vt) + t\,\eta(v,x-vt) + v\,\xi(t,x,v)]\frac{\partial}{\partial x} + \eta(v,x-vt)\frac{\partial}{\partial v} \tag{2}$$

where ξ, η and ζ are arbitrary quite smooth functions.

Let us choose $\zeta = 0$, $\xi = t^2$ and $\eta = x - v\,t$ then operator (2) is reduced to the next one:

$$\hat{X} = t^2\,\frac{\partial}{\partial t} + x\,t\,\frac{\partial}{\partial x} + (x - v\,t)\,\frac{\partial}{\partial v}\,. \tag{3}$$

Solutions of Lie's equations for operator (3) are equal to:

$$\bar{t} = \frac{t}{1 - a\,t}\,, \qquad \bar{x} = \frac{x}{1 - a\,t}\,, \qquad \bar{v} = v + a\,(x - v\,t)\,, \tag{4}$$

where a is a parameter of the Lie group (4).

It means that if $\bar{v}(x,t)$ is exact solution of equation (1) then

$$v(x,t) = -\frac{a\,x}{1 - a\,t} + \frac{1}{1 - a\,t}\,\bar{v}\left(\frac{x}{1 - a\,t}, \frac{t}{1 - a\,t}\right) \tag{5}$$

is exact solution of the Riemann equation too.

In particular if $\bar{v}(x,t)$ is solution of the Cauchy problem for equation (1) with initial condition $\bar{v}_0(x)$ then function (5) is solution of the Cauchy problem for the same equation with initial condition $v_0(x) = -a\,x + \bar{v}_0(x)$.

For instance let $\bar{v}(x,t)$ be the well-known Bessel-Fubini expansion [1] corresponding to initial condition $\bar{v}_0(x) = A_0 \sin k_0\, x$:

$$\bar{v}(x,t) = 2\,A_0 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{J_n(n\,k_0\,A_0\,t)}{n\,k_0\,A_0\,t} \sin n\,k_0\,x \tag{6}$$

where $J_n(\ldots)$ are Bessel functions of $n-$th order therefore in accordance with formula (5) the next function

$$v(x,t) = -\frac{a\,x}{1-a\,t} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,k_0\,t} J_n\left(\frac{n\,k_0\,A_0\,t}{1-a\,t}\right) \sin\left(\frac{n\,k_0\,x}{1-a\,t}\right) \tag{7}$$

obeys both the Riemann equation and initial condition $v_0(x) = -a\,x + A_0 \sin k_0\,x$.

As an another example of the approach suggested let us take the following exact solution of equation (1) [3]:

$$\bar{v}(x,t) = A_0 \exp(-\alpha\,x^2)\,[\,1 + \bar{V}(x,t)\,] \tag{8}$$

where function $\bar{V}(x,t)$ is expressed via Chebyshev-Hermite polynomials $H_n(\ldots)$:

$$\bar{V}(x,t) = \sum_{n=1}^{\infty} \frac{(\sqrt{\alpha\,(n+1)}\,A_0\,t)^n}{(n+1)!} H_n(\sqrt{\alpha\,(n+1)}\,x)\,\exp(-\alpha\,n\,x^2)\,.$$

Initial condition for solution (8) is equal to $\bar{v}_0(x) = A_0 \exp(-\alpha\,x^2)$ hence expression

$$v(x,t) = -\frac{a\,x}{1-a\,t} + A_0 \exp\left[-\frac{\alpha\,x^2}{(1-a\,t)^2}\right]\,[\,1 + V(x,t)\,] \tag{9}$$

where

$$V(x,t) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left(\frac{\sqrt{\alpha\,(n+1)}\,A_0\,t}{1-a\,t}\right)^n H_n\left(\frac{\sqrt{\alpha\,(n+1)}\,x}{1-a\,t}\right) \exp\left[\frac{-\alpha\,n\,x^2}{(1-a\,t)^2}\right]$$

is solution of the Cauchy problem for equation (1), initial condition for it being $v_0(x) = -a\,x + A_0 \exp(-\alpha\,x^2)$.

In the report presented behaviour of solutions (7) and (9) under different signs of parameter a is under consideration.

One can observe that operator (3) coincides with operator of projective group from Lie algebra of the Burgers equation [1]:

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = \mu\,\frac{\partial^2 v}{\partial x^2}\,. \tag{10}$$

Nevertheless in this circumstance there is no anything strange because under $\mu = 0$ equation (10) turns into equation (1).

In conclusion it is necessary to stress that under another choice of functions ξ, ζ and η in operator (2) one can find solution of system of Lie's equations:

$$\frac{d\bar{t}}{da} = \xi(\bar{t},\bar{x},\bar{v})\,,\ \frac{d\bar{x}}{da} = \zeta(\bar{v},\bar{x}-\bar{v}\,\bar{t}) + \bar{t}\,\eta(\bar{v},\bar{x}-\bar{v}\,\bar{t}) + \bar{v}\,\xi(\bar{t},\bar{x},\bar{v})\,,\ \frac{d\bar{v}}{da} = \eta(\bar{v},\bar{x}-\bar{v}\,\bar{t})\,,$$

$$\bar{t}(0) = t\,, \qquad \bar{x}(0) = x\,, \qquad \bar{v}(0) = v$$

which differs sharply from solution (4) and therefore another family of exact solutions of input equation starting from known solutions (6)-(9) will arise instead of formula (5).

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ABOUT AVERAGE TIME PROFIT IN STOCHASTIC MODELS OF HARVESTING A RENEWABLE RESOURCE

Rodina L. I.

Vladimir State University, Vladimir, Russia; LRodina67@mail.ru

Since 70th years of the last century problems of optimal harvesting of a resource in stochastic models was started to cause interest of scientists. One of the first works devoted the given subjects apparently is [1] in which it is shown that stochastic fish population can be harvested to achievement of certain level (escapement level) that does not dependent on the current size of population. The review of the literature devoted to questions of optimal harvesting of the populations that given by various stochastic models, is resulted in [2].

We consider models of harvesting a renewable resource given by differential equations with impulse actions, which depend on random parameters. In the absence of harvesting the population development is described by the differential equation $\dot{x} = g(x)$, which has the asymptotic stable solution $\varphi(t) \equiv K$, $K > 0$. We assume that the lengths of the intervals $\theta_k = \tau_k - \tau_{k-1}$ between the moments of impulses τ_k are random variables and the sizes of impulse action depend on random parameters v_k, where $\theta_k \in \Omega_1 \subseteq [\alpha_1, \beta_1]$, $0 < \alpha_1 \leqslant \beta_1 < \infty$, $v_k \in \Omega_2 \subseteq [0, 1]$, $k = 1, 2, \ldots$. It is possible to exert influence on the process of gathering in such a way as to stop preparation in the case where its share becomes big enough to keep some part of a resource for increasing the size of the next gathering. In this case the share of an extracted resource will be equal

$$\ell_k = \begin{cases} v_k, & \text{если} \quad v_k < u_k, \\ u_k, & \text{если} \quad v_k \geqslant u_k. \end{cases}$$

Thus, we consider the stochastic model given by the differential equation with impulse actions

$$\begin{gathered} \dot{x} = g(x), \quad t \neq \tau_k, \\ x(\tau_k) = \big(1 - \ell(v_k, u_k)\big) x(\tau_k - 0), \quad k = 1, 2, \ldots. \end{gathered} \tag{1}$$

Let’s designate $U \doteq \{\bar{u} : \bar{u} = (u_1, \ldots, u_k, \ldots)\}$, где $u_k \in [0, 1]$,

$$\bar{\theta} \doteq (\theta_1, \ldots, \theta_k, \ldots), \; \theta_k \in \Omega_1; \quad \bar{v} \doteq (v_1, \ldots, v_k, \ldots), \; v_k \in \Omega_2;$$
$$\omega_k = (\theta_k, v_k), \quad \bar{\ell} \doteq (\ell_1, \ldots, \ell_k, \ldots).$$

Let $X_k = X_k(\bar\theta, \bar\ell, x_0)$ be a quantity of a resource before gathering in the moment τ_k, $k = 1, 2, \dots$, depending on lengths of intervals $\theta_1, \dots, \theta_k$ between the moments of gathering, a resource share $\ell_i = \ell(v_i, u_i)$, $i = 1, \dots, k-1$, collected at the previous time moments and initial value x_0. For any $x_0 \geqslant 0$ we will define a limit

$$H\big(\bar\theta, \bar\ell, x_0\big) \doteq \lim_{n\to\infty} \frac{1}{\sum\limits_{k=1}^{n} \theta_k} \sum_{k=1}^{n} X_k\big(\bar\theta, \bar\ell, x_0\big)\ell_k \qquad (2)$$

which we name *average time profit* from resource extraction. If this limit does not exist, we will consider the lower limit.

We construct the control $\bar u = (u_1, \dots, u_k, \dots)$, which limits the share of an extracted resource at each time moment τ_k so that the quantity of the remaining resource, starting with some moment τ_{k_0}, is no less than a given value $x > 0$. For given control we received estimations of average time profit and conditions at which exists with probability one a positive limit (2). Thus, we describe a way of long-term extraction of a resource for the gathering mode in which some part of population necessary for its further restoration constantly remains and there is a limit of average time profit with probability one.

The stoshastic model corresponding to the differential equation with random parametres (1), is described in work [3].

Let $\varphi(t, x)$ be the solution of differential equation $\dot x = g(x)$, satisfying to the initial condition $\varphi(0, x) = x$, where $t \geqslant 0$, $x \geqslant 0$. If $\theta \in \Omega_1$, then function $\varphi(\theta, x)$ is a random variable on the set Ω_1. For any $m \in \mathbb{N}$ we will define $\sigma_m \doteq (\omega_1, \dots, \omega_m)$ and random variables $A_m = A_m(\sigma_m, x)$, $B_m = B_m(\sigma_m, x)$:

$$A_1 = \varphi(\theta_1, x), \;\; A_{k+1} = \varphi\big(\theta_{k+1}, A_k(1-\ell_k)\big);$$
$$B_1 = K, \;\; B_{k+1} = \varphi\big(\theta_{k+1}, B_k(1-\ell_k)\big), \quad k = 1, \dots, m-1.$$

Here

$$\ell_k = \ell_k(\sigma_k, x) = \begin{cases} v_k, \quad \text{если} \;\; v_k < u_k, \\ u_k, \quad \text{если} \;\; v_k \geqslant u_k = 1 - \dfrac{x}{A_k(\sigma_k, x)}, \end{cases} \qquad (3)$$

$\ell_m = \ell_m(\sigma_m, x)$ also we will define by (3). The letter M we will designate the expectation of random variable, $M\theta$ — expectation of lengths of intervals $\theta_1, \theta_2, \dots$.

Theorem 1. *Let following conditions are satisfied:*

1) *the equation* $\dot{x} = g(x)$ *has the asyptotical stable solution* $\varphi(t) \equiv K$ *and interval* (K_1, K_2) *is the area of an attraction of this solution* $(0 \leqslant K_1 < K < K_2 \leqslant +\infty)$.

2) $\Omega_1 \subseteq [\alpha_1, \beta_1]$, $\Omega_2 \subseteq [\alpha_2, \beta_2]$, *where* $0 < \alpha_1 \leqslant \beta_1 < \infty$, $0 < \alpha_2 \leqslant \beta_2 \leqslant 1$.

Then for all $m \in \mathbb{N}$, $x \in (K_1, K)$ *and* $x_0 \in (K_1, K_2)$ *there is a control* $\bar{u} \in U$ *such, that inequalities*

$$\frac{1}{m \cdot M\theta} \sum_{k=1}^{m} M(A_k \ell_k) \leqslant H_*\big(\bar{\theta}, \bar{\ell}, x_0\big) \leqslant H^*\big(\bar{\theta}, \bar{\ell}, x_0\big) \leqslant \frac{1}{m \cdot M\theta} \sum_{k=1}^{m} M(B_k \ell_k)$$

are satisfied with probability one.

Theorem 2. *Let conditions of the theorem 1 are satisfied and* $g'(x) < 0$ *for all* $x \in (L_1, L_2)$, *where* $K_1 < L_1 < K < L_2$. *Then for all* $(x, x_0) \in (L_1, K) \times (K_1, K_2)$ *at some control* $\bar{u} \in U$ *with probability one exists a positive limit*

$$H\big(\bar{\theta}, \bar{\ell}, x_0\big) = \frac{1}{M\theta} \cdot \lim_{n \to \infty} M(A_n \ell_n) = \frac{1}{M\theta} \cdot \lim_{n \to \infty} M(B_n \ell_n),$$

not dependent on initial value $x_0 \in (K_1, K_2)$.

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TOPOLOGICAL ATLAS OF INTEGRABLE PROBLEMS OF RIGID BODY DYNAMICS

Ryabov P. E.[1,2], Sokolov S. V.[2,3]

[1] *Financial University under Government of the Russian Federation, Moscow, Russia;* peryabov@fa.ru

[2] *Institute for Machines Science RAS, Moscow, Russia;*

[3] *Moscow Institute of Physics and Technology (State University), Dolgoprudny, Russia;* sokolovsv72@mail.ru

The report presents the problems and results of the study of phase topology of integrable systems with two and three degrees of freedom of the dynamics of the rigid body, supposing the Lax representation. The basis of such research was the notion of a topological atlas, introduced by M. P. Kharlamov in the early 2000s for irreducible integrable systems with three degrees of freedom [1]. The topological atlas includes an analytical description of the critical subsystems of the complete momentum map, each of which, for fixed physical parameters, is an almost Hamiltonian system with fewer degrees of freedom; a classification of equipped Smale isoenergetic diagrams with a complete description of the regular Liouville tori and their bifurcations; determination of the types of all critical points of the complete the momentum map and the creation program-constructor of topological invariants. As it turned out, to date, local and semi-local study of critical subsystems is an effective tool for the construction of rough topological invariants. For integrable Hamiltonian systems with n degrees of freedom with polynomial or rational integrals, the set of critical values of the the momentum map $\mathcal{F}$ can be written as $P = 0$, where P is a polynomial of phase variables. The decomposition of it into irreducible factors $P = \prod_j L_j$ leads to the determination of the critical subsystem of $\mathcal{M}_j$ as the set of critical points of the zero level of a function L_j. It turns out that the critical point of rank k is locally the intersection point of $n-k$ subdomains of critical subsystems. Integrals L_j of these subsystems generate symplectic operators $\mathcal{A}_{L_j}$, which determine the type of critical point. Bifurcations that occur at the intersection of $\mathcal{F}(M_j)$ surfaces at a point $\mathcal{F}(x)$, generate a semi-local type of critical point. This approach leads to an analytical description of topological invariants only in terms of the first integrals. For some integrable problems of rigid body dynamics (the Kovalevskaya top in the double force field, the Kovalevskaya-Sokolov integrable case, the Kovalevskaya-Yehia integrable case), it was possible to effectively implement the program of building a topological atlas [2], [3], [4]. The report also presents some results of constructing a topological atlas of an integrable system with three degrees of freedom on the co-algebra $e(3,2)^*$, which describes the dynamics of a generalized two-field gyrostat (the case of Sokolov-Tsyganov integrability) [5]. This is one of the most common cases of gyrostat integrability in a double field with conditions of the Kovalevskaya and gyroscopic forces with a non-constant gyroscopic moment. In general, it was not possible even to write out additional integrals in the foreseeable form [6]. To date, about 150 equipped isoenergetic diagrams of the full momentum map with indication of all cameras, families of regular 3-dimensional tori and their 4-dimensional bifurcations have been determined.

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THE GALERKIN METHOD IN THE PROBLEM OF INHOMOGENEOUS EQUILIBRIUM STATES FOR ONE OF THE BOUNDARY VALUE PROBLEMS OF THE KURAMOTO-SIVASHINSKY EQUATION

Sekatskaya A. V.

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
alinastart@mail.ru

In the work the Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} - b\frac{\partial^2 u}{\partial x^2} - c\left(\frac{\partial u}{\partial x}\right)^2, \tag{1}$$

with homogeneous Neumann boundary conditions

$$u_x(t,0) = u_x(t,\pi) = u_{xxx}(t,0) = u_{xxx}(t,\pi) = 0, \tag{2}$$

where $b \in \mathbb{R},\ u = u(t,x),\ t \leqslant 0,\ x \in [0,\pi]$ is considered. In this paper, the question of the existence of equilibrium states of the second kind on the basis of the Galerkin method was investigated. The four- and five-term Galerkin approximation is used. For the resulting system of differential equations

$$\dot{u_0} = -\left(u_1^2 + 4u_2^2 + 9u_3^2 + 16u_4^2\right), \tag{3}$$

$$\begin{gathered} \dot{u_1} = -u_1 + bu_1 - \left(6u_2u_3 + 12u_3u_4 + 2u_1u_2\right), \\ \dot{u_2} = -16u_2 + 4bu_2 + \left(\frac{u_1^2}{2} - 3u_1u_3 - 8u_2u_4\right), \\ \dot{u_3} = -81u_3 + 9bu_3 - 4u_1u_4 + 2u_1u_2, \\ \dot{u_4} = -256u_4 + 16bu_4 + 2u_2^2 + 3u_1u_3, \end{gathered} \tag{4}$$

the question of existence of equilibrium states of the second kind in the problem (1), (2) was investigated, and the question of their stability was studied. A partial comparison with the results of D. Armbruster, D. Gukenheimer, F. Holmes, as well as with the results obtained on the basis of the methods of the qualitative theory of differential equations with infinite-dimensional phase space is made. In particular, the analysis of equations for the coordinates of equilibrium states was obtained by using of numerical methods and modern computer technologies that make it possible to obtain formulas for the coordinates of equilibrium states.

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SEMICLASSICAL QUANTIZATION OF COMPLEX VECTOR BUNDLES OVER SINGULAR INVARIANT CURVES OF HAMILTONIAN SYSTEMS

Shafarevich A. I.

Moscow State University, Moscow, Russia; shafarev@yahoo.com

Semiclassical asymptotics of eigenvalues of quantum operators can be associated with invariant geometrical objects of corresponding classical Hamiltonian systems. In particular, certain sequences of eigenvalues correspond to special complex vector bundles over isotropic manifolds (Maslov complex germs). We study eigenvalues, corresponding to singular sets – namely, to vector bundles over singular invariant curves (separatrices) of partially integrable Hamiltonian systems.

INTEGRABLE DYNAMICAL SYSTEMS WITH DISSIPATION

Shamolin M. V.

Lomonosov Moscow State University, Moscow, Russian Federation;
shamolin@rambler.ru, shamolin@imec.msu.ru

We study nonconservative systems for which the usual methods of the study, e.g., Hamiltonian systems, are inapplicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics. We generalize previously known cases and obtain new cases of the complete integrability in transcendental functions of the equation of dynamics of a multi-dimensional rigid body in a nonconservative force field (see also [1]).

We obtain a series of complete integrable nonconservative dynamical systems with nontrivial symmetries. Moreover, in almost all cases, all first integrals are expressed through finite combinations of elementary functions; these first integrals are transcendental functions of their variables. In this case, the transcendence is understood in the sense of complex analysis, when the analytic continuation of a function into the complex plane has essentially singular points. This fact is caused by the existence of attracting and repelling limit sets in the system (for example, attracting and repelling focuses) (see also [2]).

Problems examined are described by dynamical systems with so-called variable dissipation with zero mean. The problem of the search for complete sets of transcendental first integrals of systems with dissipation is quite topical; a large number of works are devoted to it. Due to the existence of nontrivial symmetry groups of such systems, we can prove that these systems possess variable dissipation with zero mean, which means that on the average for a period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either the energy pumping or dissipation can occur. As applications, we study dynamical equations of motion arising in the study of the plane and spatial dynamics of a rigid body interacting with a medium and also a possible generalization of the obtained methods for the study of general systems arising in the qualitative theory of ordinary differential equations, in the theory of dynamical systems, and also in oscillation theory.

This activity is also devoted to general aspects of the integrability of dynamical systems with variable dissipation. First, we propose a descriptive characteristic of such systems. The term “variable dissipation” refers to the possibility of alternation of its sign rather than to the value of the dissipation coefficient (therefore, it is more reasonable to use the term “sign-alternating”) (see also [3]).

The assertions obtained in the work for variable dissipation system are a continuation of the Poincaré–Bendixon theory for systems on closed two-dimensional manifolds and the topological classification of such systems.

The problems considered in the work stimulate the development of qualitative tools of studying, and, therefore, in a natural way, there arises a qualitative variable dissipation system theory.

Following Poincaré, we improve some qualitative methods for finding key trajectories, i.e., the trajectories such that the global qualitative location of all other trajectories depends on the location and the topological type of these trajectories. Therefore, we can naturally pass to a complete qualitative study of the dynamical system considered in the whole phase space. We also obtain condition for existence of the bifurcation birth stable and unstable limit cycles for the systems describing the body motion in a resisting medium under the streamline flow around. We find methods for finding any closed trajectories in the phase spaces of such systems and also present criteria for the absence of any such trajectories. We extend the Poincaré topographical plane system theory and the comparison system theory to the spatial case. We study some elements of the theory of monotone vector fields on orientable surfaces.

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FULL SYMMETRIC TODA SYSTEM AND GEOMETRY OF REAL BRUHAT CELLS

Sharygin G. I.

Dept.of Mechanics and Mathematics, Moscow State University, Leninskie Gory, 1 Moscow, Russia; sharygin@itep.ru

Schubert cells in standard flag manifolds and their generalizations to arbitrary Lie groups, Bruhat cells, play important role in algebraic geometry and representation theory. These cells are open dense subsets in algebraic subvarieties in flag spaces (Schubert varieties), enumerated by the elements of the corresponding Weyl group. Namely, *the Schubert cell* $X_w^{\mathbb{C}}$*, corresponding to* $w \in W(\mathfrak{g}_{\mathbb{C}})$ is equal to the $B_{\mathbb{C}}^+$ orbit of the element $[w] \in Fl(\mathfrak{g}_{\mathbb{C}}) = G_{\mathbb{C}}/B_{\mathbb{C}}^+$. Here $[w]$ is the point in $Fl(\mathfrak{g}_{\mathbb{C}})$, represented by the element $\tilde{w}$ in the normalizer of Cartan subalgebra, corresponding to $w \in W(\mathfrak{g}_{\mathbb{C}})$. Similarly *dual Schubert cell* $Y_w^{\mathbb{C}}$*, corresponding to* w is the orbit $B_{\mathbb{C}}^- \cdot [w]$. Similarly, one defines real Bruhat cells X_w and dual real Bruhat cells Y_w in real flag manifold $Fl(G) = G/B^+ = K/Z_K(\mathfrak{h})$ (where B^+ is the real Borel subgroup) as the orbits of $[w]$ with respect to B^+ and B^-.

In complex case the structure of these cells is relatively well understood and has been extensively studied in the last 40 years. For example, using complex geometry and algebraic groups theory one can show that complex Bruhat cells intersect dual cells in accordance with Bruhat order on the corresonding Weil group: $Y_w^{\mathbb{C}} \bigcap X_{w'}^{\mathbb{C}} \neq \emptyset$ *iff* $w' \prec w$ *in Bruhat order*. Moreover, in the latter case the intersection is always transversal and the (complex) dimension of this variety is equal to the difference of *lengths* of the corresponding Weil elements: $l(w) - l(w')$.

It turned out that the similar properties for real Bruhat cells, though mutely implied by many authors, have not been accurately proved to this moment. In my talk I shall show that the following is true:

Proposition 1 *Suppose the real Lie algebra* $\mathfrak{g}$ *of* G *is non-split (normal). Then the intersection* $X_w \bigcap Y_{w'}$ *is nonempty, iff* $w' \prec w$ *in Bruhat order; moreover, if it is not empty, its (real) dimension is equal to the difference* $l(w) - l(w')$.

In addition to filling the existing gap (this statement used to be accurately proved only in the case of $G = SL_n(\mathbb{R})$, where it followed from the geometric description of Schubert cells in terms of matrix ranks), the purpose of this talk is to demonstrate the connection of this theory with the full symmetric Toda flow. Namely, the result follows from the general properties of the Toda flow on real groups. In this sense the talk paper is a continuation of the research, we began in the papers 1, 2 and 3.

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INTEGRABLE NON-ABELIAN QUADRATIC ODES

Sokolov V. V.

Landau Institute for Theoretical Physics, Chernogolovka, Russia;
Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia;
UFABC, Sao Paulo, Brasil;
vsokolov@landau.ac.ru

A classification of integrable non-abelian quadratic ODE systems with two variables is given. As the main example, an integrable non-abelian generalization of a Hamiltonian flow on an elliptic curve is presented. A Lax pair for this non-abelian system is found.

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DYNAMICS IN NEURAL NETWORKS AND INTEGRABLE SYSTEMS

Talalaev D.

Moscow State University, Russia; dtalalaev@yandex.ru

The report will focus on some problems of the modern theory of artificial neural networks and their connection with integrable structures in statistical physics. The main example of a nonlinear dynamical neural network in this talk is the Hopfield's multiattractor network [1,2]. I will describe the main problems stated in the mathematical context of this model and describe the relationship of these problems with the methods of exactly solved models of statistical mechanics, including the Ising model. The exposition is based on works [3,4].

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ON UNIVERSAL SPACE OF PARAMETERS FOR COMPACT TORUS ACTIONS ON GRASSMANNIANS

Timofeeva N. V.

Centre of Integrable Systems, P. G. Demidov Yaroslavl State University, Yaroslavl, Russia; ntimofeeva@list.ru

The complex Grassmann variety $G(k,n)$ of k-dimensional vector subspaces of n-dimensional complex vector space $\mathbb{C}^n$ is acted upon by algebraic n-torus $(\mathbb{C}^*)^n$ as

$$(\alpha_1,\ldots,\alpha_n)\star(z_1,\ldots,z_n)=(\alpha_1 z_1,\ldots,\alpha_n z_n)$$

for any $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^*)^n$. Also there is a compact torus subgroup $T^n < (\mathbb{C}^*)^n$, $T^n = \{(\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^*)^n \| |\alpha_i| = 1, \quad i = i, \ldots, n\}$ and its canonical action which is induced by its inclusion as a subgroup.

There is a moment map for $G(k,n)$ whose image is a hypersimplex $\Delta(k,n)$. This is a map $\mu : G(k,n) \to \mathbb{R}^n$ defined by the formula

$$\mu(L) = \frac{\sum_J |P^J(L)|^2 \delta_J}{\sum_J |P^J(L)|^2}$$

where P^J is Plücker coordinate, $L \in G(k,n)$, J runs over k-element subsets in $\{1, \ldots, n\}$ and $\delta_J \in \mathbb{R}^n$ is given by $(\delta_J)_i = 1$ for $i \in J$ while $(\delta_J)_i = 0$ for $i \notin J$. The moment map is equivariant under the action of the compact torus T^n.

In particular, in the case $k = 1$ we have $G(1,n) = \mathbb{P}^{n-1}$ which is a toric variety and $\Delta(1,n)$ is a simplex. For $k \geq 2$ the combinatorics of the polytope $\Delta(k,n)$ does not determine the structure of the orbit space $G(k,n)/T^n$.

Plücker embedding $Pl : G(k,n) \hookrightarrow P(\bigwedge^k \mathbb{C}^n)$ induces $(\mathbb{C}^*)^n$-invariant subdivision for $G(k,n)$ by strata W_σ where σ is the collection of Plücker coordinates which vanish along W_σ and others do not. Empty set $\sigma = \emptyset$ marks main stratum $W = W_\emptyset$. The union $\bigcup_\sigma W_\sigma/T^n$ gives the subdivision of the orbit space $G(k,n)/T^n$. The stratification by W_σ coincides with stratifications introduced by different methods by Gel'fand and Serganova, MacPherson and Goreski. The image of the moment map restricted to W_σ is the interior $\overset{\circ}{P}_\sigma$ of the admissible polytope P_σ. P_σ is the convex hull of some subset of vertices of $\Delta(k,n)$. In [1] authors prove that all points from W_σ have equal stabilizers. The orbit space $F_\sigma = W_\sigma/(\mathbb{C}^*)^n$ is an algebraic variety which is called the space of parameters for W_σ. A point from the orbit space $G(k,n)/T^n$ is determined by two coordinates: one from P_σ and another from F_σ. Also there is a subtorus $T^\sigma \leq T^n$ acting freely upon W_σ and such that $\dim T^\sigma = \dim P_\sigma$.

Let E be the disjoint union $E = \bigsqcup_\sigma (\overset{\circ}{P}_\sigma \times F_\sigma)$. Canonical projections $W_\sigma \to \overset{\circ}{P}_\sigma$ and $W_\sigma \to F_\sigma$ define the homeomorphism $W_\sigma \simeq \overset{\circ}{P}_\sigma \times F_\sigma$ [1, Theorem 4] and the bijection $E \overset{\sim}{\to} G(k,n)/T^n$. The key problem is to introduce the appropriate topology on E such that this bijection become a homeomorphism. In order to do it Buchstaber and Terzić introduce new notions: the universal space of parameters $\mathcal{F}$ which is an algebraic variety compactifying space of parameters F of the main stratum, and, for each admissible set σ, the virtual space of parameters $\widetilde{F}_\sigma$ such that there is an embedding $\widetilde{F}_\sigma \subset \mathcal{F}$ and the projection $p_\sigma : \widetilde{F}_\sigma \to F_\sigma$. Also they introduce the set $C(\Delta)$ which is formal union of admissible polytopes P_σ. This set is equipped with the topology induced by the demand for the moment map $G(k,n) \to \Delta(k,n)$ to factor through $C(\Delta)$, i.e. by the decomposition $G(k,n)/T^n \to C(\Delta) \to \Delta(k,n)$.

The set $\mathcal{E}$ defined as the union $\mathcal{E} = \bigcup_\sigma (\overset{\circ}{P}_\sigma \times \widetilde{F}_\sigma)$ inherits topology from its canonical embedding into the space $C(\Delta) \times \mathcal{F}$. The surjective map $\mathcal{E} \to E$ induces topology on E such that the orbit space $G(2,n)/T^n$ is homeomorphic to E with this topology [1, Theorem 1].

In [2] it is proven directly that the universal space of parameters for T^4 on $G(2,4)$ coincides with its space of parameters $\bigcup_\sigma F_\sigma$. It is homeomorphic to $\mathbb{CP}^1$. In [1] the universal space of parameters is obtained by means of direct computations for $G(2,5)$ and the authors proved its homeomorphicity to the smooth algebraic surface obtained from $\mathbb{CP}^2$ by blowing up its four points. In both cases $\mathcal{F}$ is homeomorphic to the Chow quotient of the corresponding Grassmannian by algebraic torus.

M. M. Kapranov in his seminal paper [3] studied Chow quotients of complex Grassmannians $G(2,n)$ under $(\mathbb{C}^*)^n$ from different points of view.

V. M. Buchstaber posed a problem: prove that universal space of parameters of compact torus T^n on $G(2,n)$ is homeomorphic to Chow quotient $G(2,n)//(\mathbb{C}^*)^n$.

I will speak about the proof for general n.

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DECOMPOSITIONS OF SURFACES FLOWS AND THEIR APPLICATIONS

Tomoo Yokoyama

Department of Mathematics, Kyoto University of Education, Japan Science and Technology Agency, Kyoto, Japan;
tomoo@kyokyo-u.ac.jp

In this talk, we describe properties of surface flows of finite type. First, we describe properties of "border points" of a flow without degenerate singular points on a compact surface. In particular, we show that each connected component of the complement of the "saddle connection diagram" is either an open disk, an open annulus, a torus, a Klein bottle, an open Möbius band, or an open essential subset. Second, we introduce a complete invariant for surface flows of "finite type". In fact, although the set of topological equivalence classes of minimal flows (resp. Denjoy flows) on a torus is uncountable, we enumerate the set of topological equivalence classes of flows with at most finitely many limit cycles but without Q-sets and non-degenerate singular points on a compact surface using three finite labelled graphs. The class of such surface flows contains both the class of Morse Smale flows (resp. area-preserving flows with finite singular points). Third, if time allows, we introduce a generalization of the Poincaré-Bendixson theorem for a flow with arbitrarily many singular points on a compact surface. In fact, the ω-limit set of any non-closed orbit is either a nowhere dense subset of singular points, a limit cycle, a limit "quasi-circuit", a locally dense Q-set, or a "quasi-Q-set".

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ON A QUANTUM HEAVY PARTICLE

Treschev D. V.

Steklov Mathematical Institute, Moscow, Russia; treschev@mi.ras.ru

We consider Schrödinger equation for a particle on a flat torus in a bounded potential, depending on time. Mass of the particle equals $1/\mu^2$, where μ is a small parameter. We show that H^1-norm of the wave function grows approximately linearly on the time interval $t \in [0, t_*]$, where t_* is slightly less than $O(1/\mu)$.

CRITERION OF COMPONENT-WISE STABILITY OF $2N$-DIMENSIONAL SADDLE SINGULARITIES IN INTEGRABLE HAMILTONIAN SYSTEMS

Tuzhilin M. A.

Lomonosov Moscow State University, Moscow, Russia;
mtu1993@mail.ru

Let we have an integrable Hamiltonian system with n degrees of freedom. The system is defined by n independent integrals in involution on $2n$-dimensional symplectic manifold (M^{2n}, ω). The map $M^{2n} \to \mathbb{R}^n$ given by the integrals is called the momentum map. The momentum map generate a foliation on a neighbourhood of a singular value so-called $2n$-dimensional singularity. Consider the singularity of a zero rank.

In the 4-dimensional case there is the Fomenko hypothesis that says if all singular points of the momentum map are non-degenerate then the foliation of the singularity is completely defined by the foliation of the singularity boundary up to Liouville equivalence. If the singular value has non-saddle type then the hypothesis is true, but if the type is saddle then it is not, see [2] and [3].

Consider the case of a saddle $2n$-dimensional singularity. The Zung theorem says that any non-splitting by Zung saddle singularity can be represented by a factor of a direct product of n 2-dimensional atoms divided by a finite group so-called almost direct product. The main problem we set is to find when this singularity is stable or non-stable. It turns out that the group action on singular zero-rank (Morse) points of each 2-atom defines the stability of almost direct product in the component-wise case. In this report we will give definitions of splitting in this case and introduce the criteria of component-wise stability.

This report requires some knowledge of the atoms and molecules theory, see [1], and the Zung theorem [4].

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THE TOPOLOGY OF THE LIOUVILLE FOLIATION OF THE ISOENERGY SURFACE OF THE SIMPLE BILLIARD BOOK

Vediushkina V. V.

MSU, Moscow, Russian Federation; arinir@yandex.ru

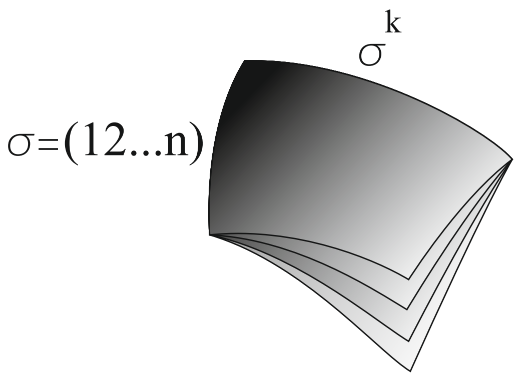
Consider a part of the plane bounded by arcs of confocal quadrics. We integrate such a billiard. The straight lines containing the segments of the polygonal billiard trajectory are tangent to a certain quadric (ellipse or hyperbola). This quadric belongs to the same class of confocal quadrics as the quadrics whose arcs form the boundary of the domain (billiard). Such billiard we call elementary billiard.

Let fix the elementary billiard Ω and a number $n \in \mathbb{N}$. To each arc of the boundary Ω, which is a connected part of the quadric, we assign an arbitrary permutation σ of order n in the following way. We require that, first, only the identical permutations are assigned to nonconvex arcs of the boundary. Secondly, in each corner of the billiard, the commuting permutations are ascribed to the arcs of the boundary forming it. Consider a disjoint union of n elementary billiards (leaves) $\Omega_i, i \in \{1..n\}$. We make from these sheets the CW-complex as follows.

Consider the arc l of the Ω billiard boundary and the permutation σ assigned to it. We expand σ into a product of independent cycles. We identify isometrically the arcs l of billiards Ω_i, if their numbers i lie in the same cycle of the permutation σ. The identified arcs will be called *spine*.

The resulting CW-complex $\mathbb{B}$ will be called *billiard book* (or briefly *book*). Generally speaking, the book $\mathbb{B}$ may be disconnected. We confine ourselves to such sets of permutations σ on the boundaries Ω, so that the book $\mathbb{B}$ is connected.

Billiard motion on the book $\mathbb{B}$ is defined as follows. Within each Ω_i sheet, the motion is rectilinear, and when it hits the boundary, a reflection occurs at which the point continues to move along the sheet $\sigma(i)$, where σ is the permutation assigned to this edge arc. Upon entering the corner, the point continues to move along the sheet $(\sigma_1 \circ \sigma_2)(i)$, where σ_1 and σ_2 are commuting permutations assigned to the sides of this angle.



Theorem. *Let B_0 be an elementary billiard which has an empty intersection with the focal line and bounded by two arcs of hyperbolas (convex and nonconvex) and two arcs of ellipses. We consider a $\mathbb{B}$ billiard book, glued together from n billiard instances B_0,, and a permutation $\sigma = (1\ 2\ ...\ n)$, a is given on the convex hyperbolic root of the book on a convex elliptic spine - permutation σ^k. Permutations corresponding to non-convex arcs of billiard B_0 are identical. Then the Fomenko-Zieschang invariant classifying the Liouville foliation of the constant-energy surface Q^3 for a given billiard book has the form $A - A$, where the label $r = \frac{k}{n}$, $\varepsilon = 1$.*

Remark. We note that in this case the isoenergy surface Q^3 is homeomorphic to the lens space $L(n, k)$.

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ABOUT ONE MATHEMATICAL MODEL IN ROTARY SYSTEM MECHANICS

Zapov A. S.

Yaroslavl State University, Yaroslavl, Russia;
yar.promo.zapov.a@gmail.com

In this paper we consider a following nonlinear boundary value problem

$$u_{tt} + (g_1 u_t + g_2 u_{t,xxxx}) + (1 - i\omega) u_{xxxx} = F, \tag{1}$$

$$u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0, x \in [0,1], \tag{2}$$

where $u = u_1(t,x) + iu_2(t,x)$ is a complex-valued function. In this case we can choose a nonlinear function F [4-6] in the following form:

$$F(u_x, u_{xx}, u_{tx}) = u_{xx}(a_1 \int_0^1 |u_x|^2 dx + a_2 \frac{d}{dt} \int_0^1 |u_x|^2 dx).$$

Finally, g_1, g_2, ω, a_1 and a_2 are some positive (non-negative in particular cases of setting targets of this problem) constants. The result parameter ω is a normalized shaft rotation speed ([1-3]). The boundary-value problem (1), (2) arises in rotary system mechanics and describes the transverse vibrations of rotating rotor of a constant cross-section from a viscoelastic material whose ends are pivotally fixed. Equation (1) is given in a renormalized form.

The linear analysis of problem (1), (2) leads to the next statement.

Theorem 1. *The solution $u \equiv 0$ is asymptotically stable, if $\omega < \omega_*$ and loses stability in case $\omega > \omega_*$. In case $\omega = \omega_*$ the critical case is realized.*

The variable ω_* is positive and defined as

$$\omega_* = min\,\omega_n,\, \omega_n = \frac{g_1 + g_2(\pi n)^4}{(\pi n)^2}, n = 1, 2, 3...$$

Also, this minimum is realized on number of m and defined as

$$\omega_* = min\,(\omega_{m_0}, \omega_{m_0+1}),\; m_0 = entier\left[\frac{1}{\pi}\sqrt[4]{\frac{g_1}{g_2}}\right].$$

Theorem 2. *The boundary-value problem (1), (2) has a non-trivial self-similar periodic solution of the form*

$$u(t, x) = \eta_n exp(i\sigma_n t) sin(\pi n x),$$

where

$$\eta_n^2 = \frac{2}{a_1\omega_n^2}(\omega^2 - \omega_n^2),\; \sigma_n = \frac{\omega(\pi n)^4}{g_1 + (\pi n)^4 g_2},$$

if $\omega > \omega_n$

Solutions with the number of $n\,(u_n(t, x))$ are asymptotically stable, if the following inequality will be correct for all natural values of k:

$$\left(1 - \frac{\omega^2}{\omega_k^2}\right)(\pi k)^2 + \left(\frac{\omega^2}{\omega_n^2} - 1\right)(\pi n)^2 > 0. \quad (3)$$

The stable condition (3) can be simplified in some particular cases. F.e, if we consider the solution $u_1(t, x)$ and $\omega_1 < \omega_2 < \omega_3...$, then this solution asymptotically stable at $\omega \in (\omega_1, \omega_2]$. Stability conditions analysis of equation (3) will be based on numerical analysis of conditions (3) in common case (arbitrary values of n).

Finally, zeroth solution of boundary-value problem (1), (2) is always unstable at $g_2 = 0$.

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ОЦЕНКА ИНВАРИАНТНЫХ ЧИСЛОВЫХ ПОКАЗАТЕЛЕЙ КВАЗИУСТОЙЧИВЫХ АТТРАКТОРОВ ДИНАМИЧЕСКИХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

ESTIMATION OF INVARIANT NUMERICAL EXPONENTS FOR QUASI-STABLE ATTRACTORS OF DYNAMICAL SYSTEMS WITH TIME DELAY

Алешин С. В.[1], Горюнов В. Е.[2]

[1]*ЯрГУ им. П.Г. Демидова, Ярославль, Россия; НЦЧ РАН, Черноголовка, Россия;* fktiby@yandex.ru

[2]*НЦЧ РАН, Черноголовка, Россия;* salkar@ya.ru

Мы рассматриваем задачу оценки показателей Ляпунова для систем дифференциальных уравнений с запаздыванием. Теорема Оселедеца, позволяющая эффективно вычислять показатели Ляпунова в конечномерном случае, когда линеаризованная на аттракторе система обыкновенных дифференциальных уравнений всегда является правильной по Ляпунову, и, тем самым, верхний предел может быть заменен на обычный, для таких систем, вообще говоря, не работает. Тем не менее, с помощью специальных методов удается вычислять инвариантные характеристики, качественно близкие к искомым. Предлагается новая реализация разработанного ранее алгоритма с использованием дискретного преобразования Фурье и задания особых начальных условий для линеаризованных на аттракторе динамических систем.

Для целого ряда моделей генных сетей и нейронных ассоциаций, исследуемых в последнее время, является характерным так называемое квазиустойчивое поведение. Феномен квазиустойчивости цикла (k-мерного тора) динамической системы заключается в том, что часть его мультипликаторов асимптотически близка к единичной окружности, а остальные мультипликаторы по модулю меньше единицы (за исключением простого единичного (k единичных)). В некоторых случаях с применением методов большого параметра удается доказать существование и дать асимптотическую оценку мультипликаторов исследуемой системы. Однако в случае, если методы большого параметра неприменимы, необходимо получить инструмент численной оценки мультипликаторов. Такой инструмент дают алгоритмы оценки показателей Ляпунова. Учитывая, что в моделях нейронных и генных сетей часто применяются уравнения с запаздыванием, алгоритм оценки показателей Ляпунова для таких систем будет широко востребован.

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МОДЕЛЬ САЛЬТАТОРНОГО ПРОВЕДЕНИЯ ПОСЛЕДОВАТЕЛЬНОСТИ ИМПУЛЬСОВ НА ОСНОВЕ МОДИФИЦИРОВАННОГО ИМПУЛЬСНОГО НЕЙРОНА

THE SALTATORY CONDUCTION OF A PULSE SEQUENCE MODEL BASED ON THE MODIFIED PULSE NEURON

Ануфриенко С. Е.[1], **Борисов А. А.**[2]

[1] *Ярославский государственный университет им. П.Г. Демидова, г. Ярославль, Россия;* sanufrienko@rambler.ru

[2] *Ярославский государственный университет им. П.Г. Демидова, г. Ярославль, Россия;* borisov.alex@yandex.ru

На основе многочисленных исследований [1],[2] В.В. Майоровым была предложена модель проведения импульсов по миелинизированным волокнам, основанная на разработанной им же модели импульсного нейрона [3]. В данной работе используются модифицированные модели импульсных нейронов как порогового, так и автогенератора.

Уравнение, описывающее мембранный потенциал модифицированного порогового нейрона имеет вид:

$$\dot{u} = \lambda[(a - f^1_{Na}(u))f^2_{Na}(u(t-1)) - 1 + f_K(u(t-1))]u + \varepsilon f_l(u(t-1)).$$

Здесь $\lambda \gg 1$ отображает большую скорость распространения электрохимических процессов, $0 < \varepsilon \ll 1$ - параметр, описывающий токи утечки.

Функции $f^1_{Na}(u), f^2_{Na}(u(t-1)), f_K(u(t-1)), f_l(u)$ описывают состояние каналов ионов натрия, калия и токов утечки соответственно. Они достаточно гладкие, положительные.

$f^1_{Na}(u), f^2_{Na}(u(t-1)), f_K(u(t-1))$ при $u \to \infty$ стремятся к нулю быстрее, чем $O(u^{-1})$. $f^2_{Na}(0) = 1$.

$f_l(u)$ при $u \to \infty$ стремится к нулю быстрее, чем $O(u^{-2})$. $f_l(0) = 1$.

Модель нейрона-автогенератора выглядит следующим образом:

$$\dot{u} = \lambda[(a - f^{10}_{Na}(u))f^{20}_{Na}(u(t-1)) - 1 + f^0_K(u(t-1))]u.$$

Свойства функций $f^{10}_{Na}(u), f^{20}_{Na}(u(t-1)), f^0_K(u(t-1))$ аналогичны свойствам соответствующих функций для порогового нейрона.

Система, описывающая процесс распространения последовательности импульсов по аксону имеет вид:

$$\dot{u}_0 = \lambda[(a - f_{Na}^{10}(u_0))f_{Na}^{20}(u_0(t-1)) - 1 + f_K^0(u_0(t-1))]u_0 + e^{-\lambda\sigma}(v_1 - u_0), \tag{1}$$

$$\dot{u}_i = \lambda[(a - f_{Na}^1(u_i))f_{Na}^2(u_i(t-1)) - 1 + f_K(u_i(t-1))]u_i + \\ + \varepsilon f_l(u_i(t-1)) + e^{-\lambda\sigma}(v_i - 2u_i + v_{i+1}), \tag{2}$$

$$i = 1, \dots, N-1;$$

$$\dot{u}_N = \lambda[(a - f_{Na}^1(u_N))f_{Na}^2(u_N(t-1)) - 1 + f_K(u_N(t-1))]u_N + \\ + \varepsilon f_l(u_N(t-1)) + e^{-\lambda\sigma}(v_N - u_N), \tag{3}$$

$$\dot{v}_i = \lambda(u_{i-1} - 2v_i + u_i), \quad i = 1, \dots, N. \tag{4}$$

Введем параметры:

$$\alpha^0 = a - f_{Na}^{10}(0) - 1 - f_K^0(0) > 0,$$
$$\alpha_1^0 = f_K^0(0) - 1 > 1,$$
$$\alpha = f_{Na}^1(0) - a - f_K(0) + 1 > 0,$$
$$\alpha_1 = f_K(0) - 1 > 1,$$
$$0 < \sigma < \alpha_1.$$

Применяя метод пошагового асимптотического интегрирования уравнений (1)–(4) при специальных начальных условиях, получим формулы, задающие мембранный потенциал нулевого перехвата:

$$u_0(t) = \begin{cases} e^{\lambda\alpha_1^0(t+o(1))} & \text{при} \quad t \in [\delta, 1-\delta], \\ e^{\lambda(\alpha_1^0-(t-1)+o(1))} & \text{при} \quad t \in [1+\delta, 2+\alpha_1^0-\delta], \\ e^{\lambda(\alpha^0(t-\alpha_1^0-2)-1+o(1))} & \text{при} \quad t \in [2+\alpha_1^0+\delta, T-\delta]. \end{cases} \tag{5}$$

Потенциал мембраны первого перехвата Ранвье задается формулами:

$$u_1(t) = \begin{cases} e^{\lambda\alpha_1(t-\tau+o(1))} & \text{при} \quad t \in [\tau+\delta, 1+\tau-\delta], \\ e^{\lambda(\alpha_1-(t-\tau-1)+o(1))} & \text{при} \quad t \in [1+\tau+\delta, 2+\tau+\alpha_1-\delta], \\ \dfrac{\varepsilon+o(1)}{\lambda\alpha} & \text{при} \quad t > 2+\tau+\alpha_1+\delta. \end{cases} \tag{6}$$

Потенциал мембраны первого миелинизированного слоя задается формулами:

$$v_1(t) = \begin{cases} e^{\lambda\alpha_1^0(t+o(1))} & \text{при} \quad t \in [\delta, 1-\delta], \\ e^{\lambda(\alpha_1^0-(t-1)+o(1))} & \text{при} \quad t \in [1+\delta, t_*-\delta], \\ e^{\lambda\alpha_1(t-\tau+o(1))} & \text{при} \quad t \in [t_*+\delta, 1+\tau-\delta], \\ e^{\lambda(\alpha_1-(t-\tau-1)+o(1))} & \text{при} \quad t \in [1+\tau+\delta, 2+\tau+\alpha_1-\delta], \\ \dfrac{\varepsilon+o(1)}{\lambda\alpha} & \text{при} \quad t > 2+\tau+\alpha_1+\delta. \end{cases} \tag{7}$$

Здесь

$$\tau = \frac{\sigma}{\alpha_1} + o(1) < 1, \; t_* = 1 + \frac{\alpha_1\tau}{\alpha_1+1}, \; T = 2 + \alpha_1^0 + \frac{1}{\alpha^0}.$$

Формулы (5)–(7) описывают состояние первого перехвата и первого миелинизированного слоя до нового спайка нулевого нейрона. Потенциалы мембран перехватов и миелинизированных слоев с номером i задаются формулами:

$$\begin{cases} u_i(t) = u_1(t-(i-1)\tau) & \text{при} \quad t > (i-1)\tau, i = 2, ..., N \\ v_i(t) = v_1(t-(i-1)\tau) & \text{при} \quad t > (i-1)\tau, i = 2, ..., N \end{cases} \tag{8}$$

При $t = T$ нулевой перехват генерирует новый спайк. По перехватам начнет распространяться новая волна импульсов в направлении возрастания их номеров. Потенциалы мембран перехватов Ранвье и миелинизированных слоев задается формулами (5)–(8) со сдвигом по времени на величину T. Этот процесс будет повторяться с периодичностью T.

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МЕТОД ПРЕОБРАЗОВАНИЯ ФУРЬЕ ДЛЯ НЕКОТОРЫХ ТИПОВ ЛИНЕЙНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ С ПЕРЕМЕННЫМИ КОЭФФИЦИЕНТАМИ

FOURIER TRANSFORM METHOD FOR SOME TYPES OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Гишларкаев В. И.

Чеченский государственный университет, Грозный, Россия;
vakhag@mail.ru

Работа в основном посвящена исследованию задачи Коши для уравнений вида

$$\partial_t u(t,x) + \sum_{|a|\le m} \epsilon_\alpha a_\alpha(t) \partial_x^\alpha u(t,x) = f(t,x), \tag{1}$$

где $\epsilon_\alpha = x_i$ или 1. Кроме этого рассмотрен схематично подход в случае, когда вместо $\epsilon_\alpha a_\alpha(t)$ в (1) стоит $\phi_\alpha(t,x)$.

Уравнения типа (1) встречаются во многих прикладных вопросах: диффузионных процессах при наличии сноса в каком-то направлении, при описании роста популяций, в различных случаях теории тепло-и-массообмена с тепловыделением или другими дополнительными условиями (см., например, [1]). Основой для нашего анализа рассматриваемых задач служит преобразование Фурье. Модификация стандартного метода применения преобразования Фурье идет в двух направлениях:

1. Использование идеи, подсказанной схемой вывода уравнения Хопфа из статистичекой механики, описывающего динамику характеристического функционала распределения скоростей движения жидкости. А именно, ставится вопрос, для каких типов уравнений в частных производных решения задачи Коши можно представить в виде преобразования Фурье от распределения сдвига начального условия по траекториям какого-то обыкновенного дифференциального уравнения, зависящего от исходного, или системы таких уравнений при заданном распределении начального условия. Этот подход осуществлен в [2].

2. Применим к (1) обратное преобразование Фурье $x \to \xi$. Учитывая, что $F^{-1}(x_i\partial_x^{\alpha}u(t,x)) = i^{|\alpha|+1}\partial_{\xi}(\xi^{\alpha}u(t,\xi))$, получим комплексное дифференциальное уравнение 1-го порядка или, что эквивалентно, систему из двух уравнений 1-го порядка, которую можно решить методом характеристик. В случае, когда в (1) $\epsilon_{\alpha}a_{\alpha}(t)$ стоит $\phi_{\alpha}(t,x)$, то после взятия обратного преобразования Фурье от (1), получим интегро-дифференциальное уравнение, в котором дифференцирование проводится по временной переменной, а интегрирование – только по пространственным переменным. Таким образом, полученное уравнение подпадает под теорию обыкновенных дифференциальных уравнений в банаховом пространстве.

   Отметим также, что для вывода теорем существования и единственности решений рассматриваемых задач необходимо сделать удачный выбор пространства начальных данных и пространства решений, учитывающих специфику описанных выше подходов. Выбор их определяется теоремой Пэли-Винера-Шварца о преобразовании Фурье обобщенных функций с компактным носителем.

   Сформулируем теперь некоторые результаты. Введем пространство начальных данных: $C^A(\mathbb{R}^n) := \{\Phi|_{\mathbb{R}^n} : (\Phi : \mathbb{C}^n \to \mathbb{C}$ целая функция : $((\mathrm{Im}\Phi(x) = 0, \forall x \in \mathbb{R}^n) \wedge (\exists c, m, r \in \mathbb{R} : |\Phi(z)| < c(1+||z||_{\mathbb{C}^n})^m e^{r||\mathrm{Im} z||_{\mathbb{R}^n}} \forall z \in \mathbb{C}^n)))\}$ и пространство решений: $C_T^{1,A}(\mathbb{R}^n) = \{\Phi|_{[0,T]\times\mathbb{R}^n} | (\Phi(\cdot,x) \in \mathbb{C}^1(0,T) \forall x \in \mathbb{R}^n) \wedge (\Phi(t,\cdot) : \mathbb{C}^n \to C$ – целая функция : $((\mathrm{Im}\Phi(t,x) = 0\, \forall (t,x) \in [0,T] \times \mathbb{R}^n) \wedge (\exists c, m, z \in \mathbb{R} : |\Phi(z)| < c(1+||z||_{\mathbb{C}^n})^m e^{r|\mathrm{Im} z| \forall (t,z) \in [0,T]\times\mathbb{C}^n}))\}$. Для краткости ограничимся случаем $n = 1$, в общем случае формулировки аналогичны. Через $\mathbb{C}^A$, $\mathbb{C}_T^{1,A}$ обозначим $\mathbb{C}^A(\mathbb{R}^1)$, $\mathbb{C}_T^{1,A}(\mathbb{R}^1)$, соответственно.

Теорема 1. *Найдется $T > 0$ такое, что в пространстве $\mathbb{C}_T^{1,A}$ существует единственное решение задачи.*

$$\partial_t u(t,x) + x\sum_{k=0}^{n} a_{2k+1}(t)\partial_x^{2k+1}u = 0, \quad (t,x) \in (0,T)\times R, \;\; u|_{t=0} = u_0 \in \mathbb{C}^A.$$

Получены формулы, представляющие решения задачи Коши с начальными данными из $\mathbb{C}^A$ для уравнений:

$$\partial_t u(t,x) + x\sum_{k=0}^{n} a_k(t)\partial_x^k u = 0;$$

$$\partial_t u(t,x) + \sum_{k=0}^{n} a_k(t)\partial_x^k (xu) = 0;$$

$$\partial_t u(t,x) + x\sum_{k=0}^{m} a_i(t)\partial_x^i u + \sum_{i=1}^{k} b_i(t)\partial_{t,x}^{1,i} u = 0;$$

$$\partial_t u(t,x) - l(-1)^{\frac{l}{2}}\sum_{k=0}^{m} a_k(t)\partial_x^{k+l-1} u + x\sum_{k=0}^{m} a_k(t)\partial_x^k u = 0;$$

$$\partial_t u = f'(t)(-1)^{\frac{l}{2}}\partial_x^l u - l(-1)^{\frac{l}{2}} f(t)\sum_{k=0}^{m} a_k(t)\partial_x^{k+l-1} u - x\sum_{k=0}^{m} a_k(t)\partial_x^k u;$$

$$\partial_t u = \sum_{k=1}^{N} f_k'(t)(-1)^{\frac{l_k}{2}}\partial_x^{l_k} u - a(t)\sum_{k=1}^{N} l_k(-1)^{\frac{l_k}{2}} f_k(t)\partial_x^{m+l_k-1} u - xa(t)\partial_x^m u;$$

$$\partial_t u(t,x) + \sum_{k=0}^{m} b_{2k}(t)\partial_t\partial_x^{2k} u(t,x) + \sum_{k=0}^{m} a_{2k}(t)\partial_x^{2k} u(t,x) + \sum_{k=0}^{m} xa_{2k+1}(t)\partial_x^{2k+1} u(t,x) = f(t,x);$$

$$\partial_t u(t,x) + xa(t)(Pu)(t,x) = 0,$$

где $(Pu)(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}_y}\left(\int_{\mathbb{R}_\xi} e^{i(y-x)\xi}\phi(\xi)d\xi\right)u(t,y)dy$ – псевдодифференциальный оператор.

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ЛОКАЛЬНЫЕ БЫСТРО ОСЦИЛЛИРУЮЩИЕ РЕШЕНИЯ В ЗАДАЧЕ О ДИСЛОКАЦИЯХ

LOCAL RAPIDLY OSCILLATING SOLUTIONS OF THE DISLOCATION PROBLEM

Глызин С. Д.[1], Кащенко С. А.[2], Толбей А. О.[3]

[1]*Ярославский государственный университет им. П.Г. Демидова, г. Ярославль, Россия;* glyzin@uniyar.ac.ru

[2]*Ярославский государственный университет им. П.Г. Демидова, г. Ярославль, Россия;*
Национальный исследовательский ядерный университет «МИФИ», г. Москва, Россия; kasch@uniyar.ac.ru

[3]*Ярославский государственный университет им. П.Г. Демидова, г. Ярославль, Россия;* a.tolbey@uniyar.ac.ru

Рассматривается ставшее классическим нелинейное дифференциальное уравнение с отклонениями пространственной переменной, называемое задачей о дислокациях [1]

$$m\frac{d^2y_i}{dt^2} = F_{i+1,i} - F_{i,i-1} - f_0 \sin\left(\frac{2\pi y_i}{a}\right), \quad (i = 1, \dots, N) \tag{1}$$

в случае

$$F_{i+1,i} = \gamma(y_{i+1} - y_i) + \alpha(y_{i+1} - y_i)^2 + \beta(y_{i+1} - y_i)^3. \tag{2}$$

Изучаются быстро осциллирующие решения (см. также [2]), возникающие в модели (1), (2), где вместо функции $\sin y$ в правой части (1) выбрано более общее выражение $ay(t,x) + by^3(t,x)$, причем предполагается, что $a = -a_0 + \varepsilon^2 a_1, a_0 \geq 0$, f_0, a, γ — положительные коэффициенты, $y_i(t) = y(t, x_i)$ и расстояния между соседними точками x_i равны h, как и в задаче Ферми-Паста-Улама [2], [3]. Считается, что значения x_i распределены на отрезке длины $2\pi L$ и выполнено условие периодичности $y(t, x_i + 2\pi L) = y(t, x_i)$. Соотношение $x_{i+1} = x_i + \varepsilon$, где $\varepsilon = hL^{-1}$, получается после нормировки пространственной переменной $x : x \to Lx$. Далее, нормируя время $t \to (km^{-1})^{1/2}t$, выписываем краевую задачу

$$\frac{d^2y}{dt^2} = y(t,x+\varepsilon) - 2y(t,x) + y(t,x-\varepsilon) + \alpha\Big(y^2(t,x+\varepsilon) - 2y(t,x+\varepsilon) \times y(t,x) + 2y(t,x) \times y(t,x-\varepsilon) - y^2(t,x-\varepsilon)\Big) + \beta\Big((y(t,x+\varepsilon) - y(t,x))^3 - (y(t,x) - y(t,x-\varepsilon))^3\Big) + ay(t,x) + by^3(t,x), \tag{3}$$

$$y(t, x+2\pi) \equiv y(t, x). \tag{4}$$

В докладе предлагается рассмотреть вопрос о нерегулярных решениях (3), (4), формирующихся на асимптотически высоких (при $\varepsilon \to 0$) модах.

Фиксируем произвольно параметр $\delta \neq \pi n$ $(n = 0, \pm 1, \pm 2, \ldots)$ и будем исследовать решения (3), (4), формирующиеся на модах с номерами

$$k = \pm(2\delta\varepsilon^{-1} + \theta + m). \tag{5}$$

Здесь $\theta = \theta(\varepsilon) \in [0, 1)$ дополняет предыдущее в (5) слагаемое до целого, а $m = 0, \pm 1, \pm 2, \ldots$ При таких номерах k для корней $\lambda = \lambda_m$ в (5) имеем представление

$$\lambda_m = \pm i\Big[\gamma(\delta)\Big(1 + \frac{\varepsilon(\theta + m)\sin(2\delta)}{2\gamma^2(\delta)}\Big) + o(\varepsilon^2)\Big]. \tag{6}$$

Введем в рассмотрение формальный ряд

$$y = \varepsilon\Big(\sum_{m=-\infty}^{+\infty} \xi_m(\tau)\exp\Big(i\Big(\frac{2\delta}{\varepsilon} + \theta + m\Big)x + i\gamma(\delta)\Big(1 + \frac{\varepsilon(\theta+m)\sin 2\delta}{2\gamma^2(\delta)} + o(\varepsilon^2)\Big)t\Big) +$$
$$+ \sum_{m=-\infty}^{+\infty} \eta_m(\tau)\exp\Big(i\Big(\frac{2\delta}{\varepsilon} + \theta + m\Big)x - i\gamma(\delta)\Big(1 + \frac{\varepsilon(\theta+m)\sin 2\delta}{2\gamma^2(\delta)} + o(\varepsilon^2)\Big)t\Big) + \overline{cc}\Big) + \tag{7}$$
$$+\varepsilon^2 y_2(\tau, t, x) + \varepsilon^3 y_3(\tau, x, t) + \ldots$$

Здесь $\tau = \varepsilon^2 t$, через $\overline{cc}$ обозначаются слагаемые, комплексно сопряженные к содержащимся в той же скобке, а y_i – периодически зависят от x и от t. Обозначим

$$\xi(\tau, x) = \sum_{m=-\infty}^{+\infty} \xi_m(\tau)\exp(imx),\ \eta(\tau, x) = \sum_{m=-\infty}^{+\infty} \eta_m(\tau)\exp(imx).$$

Подстановка (7) в (3) на каждом шаге алгоритма дает соответствующие краевые задачи. При ε^3 получается задача для $y_3(t, \tau, x)$. Для данной задачи условия существования ограниченных решений выражаются следующей системой уравнений относительно ξ и η:

$$L^{+}(\delta)\xi = 3\xi(|\xi|^2 + 2|\eta|^2)\cdot\Big(\Big[-6 + 2\cos(2\delta) - \frac{\cos(4\delta)}{2}\Big]\cdot\beta + b\Big),$$
$$L^{-}(\delta)\eta = 3\eta(2|\xi|^2 + |\eta|^2)\cdot\Big(\Big[-6 + 2\cos(2\delta) - \frac{\cos(4\delta)}{2}\Big]\cdot\beta + b\Big) \quad (8)$$

с периодическими краевыми условиями

$$\xi(\tau, z_+ + 2\pi) \equiv \xi(\tau, z),\ \eta(\tau, z_- + 2\pi) \equiv \eta(\tau, z). \quad (9)$$

Здесь

$$L^{+}(\delta)\xi \overset{\text{def}}{=} 2i\gamma(\delta)\frac{\partial\xi}{\partial\tau} - a_1\xi - R(\delta)\cdot\Big[\frac{\partial^2\xi}{\partial z_+^2} + 2i\theta\frac{\partial\xi}{\partial z_+} - \theta^2\xi\Big],$$
$$L^{-}(\delta)\eta \overset{\text{def}}{=} -2i\gamma(\delta)\frac{\partial\eta}{\partial\tau} - a_1\eta - R(\delta)\cdot\Big[\frac{\partial^2\eta}{\partial z_-^2} + 2i\theta\frac{\partial\eta}{\partial z_-} - \theta^2\eta\Big],$$

где $R(\delta) = \cos(2\delta) - \frac{1}{4}\gamma^{-2}(\delta)\sin^2(2\delta)$ или $R(\delta) = \frac{7}{4}\cos^2(\delta) - 1$.

Основным результатом работы является построение многопараметрических семейств нелинейных систем уравнений специального вида, которые играют роль нормальных форм для изучения решений с начальными условиями из малой окрестности состояния равновесия. В частности, приведены системы нелинейных уравнений шредингеровского типа (8), (9), решениям которых отвечают быстро осциллирующие решения исходной задачи. Отметим, что в [4] обоснована возможность получения с помощью предложенной методики существенно более сложных нормализованных систем.

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О НЕКОТОРЫХ ПОДХОДАХ К РЕШЕНИЮ ЗАДАЧИ «USEFUL PROOF-OF-WORK FOR BLOCKCHAINS»

ABOUT SOME APPROACHES TO THE SOLUTION OF THE PROBLEM USEFUL PROOF-OF-WORK FOR BLOCKCHAINS

Дурнев В. Г.[1], **Мурин Д. М.**[2]

[1]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* Durnev@uniyar.ac.ru

[2]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* nirum87@mail.ru

Технология блокчейн существует уже 10 лет и основывается на ранее известных технологиях пиринговых сетей, распределенных баз данных, электронной подписи и на принципе доказательства работой «Proof-of-Work» [1]. Суть последнего принципа состоит в том, что некоторое событие (например, перевод денежных средств с одного счета на другой) становится значимым (принимается большинством участников блокчейн-сети как свершившийся факт) только после того, как оно подтверждено определенным объемом вычислительной работы.

Задачи поиска битовой строки с хэшем, удовлетворяющим определенным условиям (хэш-головоломки), которые в настоящее время широко используются для обеспечения доказательства работой, обладают существенным недостатком – у них нет никакого полезного применения за пределами технологии блокчейн.

Мы считаем, что индивидуальные представители NP-полных Задач (далее индивидуальных представителей NP-полных Задач мы будем для краткости называть задачами, соответственно Задача — это множество задач) могут считаться полезными и часто встречаются на практике [2, 3], и предлагаем подход, позволяющий использовать эти задачи для обеспечения доказательства работой.

В основе нашего подхода лежат представленные ниже ответы на следующее вопросы:

1. Как создавать задачи?
2. Каким алгоритмом решать задачи?
3. Как управлять сложностью создаваемых задач?
4. Как осуществить привязку задачи к информации блока?

Рассмотрим два основных способа создания задач. Первый состоит в сборе и использовании задач, возникающих в практической деятельности. Полезность задач при данном способе создания очевидна, но не каждая полученная таким образом задача подойдет для обеспечения доказательства работой. Задача может оказаться слишком легкой или слишком трудной для блокчейн-сети, что приводит к необходимости оценки сложности задач и ее соотнесения с вычислительными возможностями блокчейн-сети.

Другим способом создания задач является применение специализированных генераторов, которые могут создавать задачи с определенными параметрами (в том числе сложностью, размером и т. д.), согласованными с возможностями блокчейн-сети. Однако полезность создаваемых задач в общем случае не гарантирована.

Можно рассматривать также гибридные варианты создания задач, но при большом числе полезных задач применение генераторов нецелесообразно, поскольку будет отвлекать вычислительные ресурсы на искусственно созданные задачи, а при малом числе полезных задач не ясно преимущество использования генераторов перед использованием хэш-головоломок. Тем не менее использование гибридов может быть оправдано, например, в случаях неравномерного прихода полезных задач для усреднения числа задач доступных к решению в единицу времени.

Хорошо известно, что любая Задача из класса NP, может быть полиномиально сведена к Задаче КНФ-выполнимости (SAT) [4]. Следовательно, любой индивидуальный представитель NP-полной Задачи может быть сведен к индивидуальному представителю Задачи КНФ-выполнимости. Для решения задачи в форме КНФ-выполнимости можно применять SAT- решатели, например, CDSL SAT-решатели [5].

Управлять сложностью задач можно путем проверки (для первого способа создания) или задания (для второго способа создания) следующих параметров: принадлежность к особому виду, размерность, требуемая точность решения. Кроме того, существенное значение имеет практика решения задачи (в случае, если задача, которую блокчейн-сеть не смогла решить, может быть повторно предложена для решения) и близких по характеристикам задач блокчейн-сетью. Как уже упоминалось ранее, созданные задачи должны пройти отбор на предмет практической возможности решения блокчейн-сетью с заданными характеристиками.

Привязка задачи к информации блока может быть реализована несколькими способами, рассмотрим только один из них, основанный на традиционной технологии блокчейн и использующий хэш-головоломки. Мы можем включить описание задачи в дерево Меркла, а ее решение в служебную информацию блока, после чего, в соответствии с традиционной технологией блокчейн, решить хэш-головоломку для блока в такой конфигурации. Решение хэш-головоломки обеспечит связывание служебной информации блока, информации о событиях, которые необходимо подтвердить вычислительной работой, и информации о задаче, обеспечивающей события вычислительной работой. Блок, полученный таким образом, можно назвать «тяжелым», поскольку на его формирование необходимо потратить больше вычислительных ресурсов за счет решения прикрепленной к блоку задачи, чем на обычный «легкий» блок. Поскольку на формирование тяжелого блока должно тратиться больше вычислительных ресурсов, то необходимо мотивировать участников блокчейн-сети к созданию таких блоков. Этого можно добиться увеличением вознаграждения за формирование тяжелых блоков и изменением приоритета в выборе блокчейн-сетью продолжаемой блокчейн-цепочки в пользу не самой длинной цепочки, но самой длинной цепочки с учетом «веса» входящих в нее тяжелых блоков. А именно тяжелому блоку можно присвоить «вес» равный, например, трем «весам» легких блоков. Соответственно, за генерацию тяжелого блока должно осуществляться трехкратное вознаграждение, и цепочка из одного тяжелого блока должна быть эквивалентна цепочке из трех легких блоков.

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О НЕКОТОРЫХ РЕЗУЛЬТАТАХ ИССЛЕДОВАНИЙ ЗАДАЧИ 0-1-РЮКЗАК

ABOUT SOME RESULTS RESEARCH 0-1-KNAPSACK PROBLEM

Дурнев В. Г.[1]**, Мурин Д. М.**[2]

[1]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* Durnev@uniyar.ac.ru

[2]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* nirum87@mail.ru

Хорошо известно [1, 2, 3], что **Проблема «0-1-рюкзак»** является **NP**-полной и может быть сформулирована как вопрос о разрешимости в числах 0-1 уравнения вида

$$a_1x_1 + ... + a_nx_n = b,$$

где a_1, ..., a_n и b – произвольные натуральные числа, x_1, ..., x_n – неизвестные.

В 1978 году Р. Меркль и М. Хеллман [4] предложили первую (опубликованную в открытых источниках) асимметричную систему шифрования, которая базировалась на **Проблеме «0-1-рюкзак»**.

Затем в 1979-82 годах А. Шамир [5, 6] показал, что метод генерации открытых ключей по закрытым в криптосистеме Меркля-Хеллмана не является надежным, и «практически все» индивидуальные представители **Проблемы «0-1-рюкзак»**, возникающие в этой системе, поддаются решению.

Тем не менее **Проблема «0-1-рюкзак»** обладает полезными для криптографии свойствами. Во-первых, о ней известно, что она является достаточно трудной, и во-вторых, представляется вероятным, что на ее основе можно построить стойкие криптосистемы с высокой скоростью шифрования и расшифрования (например, скорость шифрования и расшифрования в криптосистеме Меркля-Хелмана существенно выше, чем в криптосистеме RSA).

В криптосистеме Меркля-Хеллмана множеством открытых ключей является множество инъективных наборов натуральных чисел, для которых **Проблема «0-1-рюкзак»** имеет не более одного решения. Поэтому изучение инъективных наборов натуральных чисел представляет особый интерес.

ОПРЕДЕЛЕНИЕ. Будем называть набор натуральных чисел $a_1, \ldots, a_n$ *инъективным* (соответственно m-*инъективным*), если не существует двух различных наборов чисел 0-1 (соответственно $0, \ldots, m-1$, где натуральное число $m \geq 2$) $\alpha_1, \ldots, \alpha_n$ и $\beta_1, \ldots, \beta_n$ таких, что

$$a_1\alpha_1 + \ldots + a_n\alpha_n = a_1\beta_1 + \ldots + a_n\beta_n.$$

В противном случае набор натуральных чисел $a_1, \ldots, a_n$ будем называть *неинъективным* (соответственно m-*неинъективным*).

Следующие теоремы дают представление о числе инъективных и m-инъективных наборов из n натуральных чисел с максимальным элементом равным M.

Теорема 1. *Число инъективных и m-инъективных наборов из n натуральных чисел с максимальным элементом равным M ограничено сверху величиной*

$$P_1(m, n, M) = n! C_{M-1}^{n-1},$$

где полагаем $M \geq n$, C_r^s – число сочетаний.

Теорема 2. *Число m-инъективных наборов из n натуральных чисел с максимальным элементом равным M ограничено снизу величиной*

$$P_2(m, n, M) = n! \Big(\frac{M - m^{n-1} + 1}{(m-1)m^{n-2}} - 1 \Big) \Big(\frac{M - m^{n-1} + 1}{m^{n-1}} \Big)^{n-2},$$

где полагаем $M \geq m^{n-1}$.

Отметим, что и $P_1(m, n, M)$, и $P_2(m, n, M)$ при фиксированном m являются полиномами $n-1$ степени от M.

Следствие 1. *Число инъективных наборов из n натуральных чисел с максимальным элементом равным M ограничено снизу величиной*

$$n! \Big(\frac{M - 2^{n-1} + 1}{2^{n-2}} - 1 \Big) \Big(\frac{M - 2^{n-1} + 1}{2^{n-1}} \Big)^{n-2}, \text{ где полагаем } M \geq 2^{n-1}.$$

Рассмотрим следующие задачи.

Задача «Инъективный рюкзак» («m-инъективный рюкзак»).

Дано: *набор натуральных чисел $a_1, \ldots, a_n$.*

Определить: *является ли набор инъективным (m-инъективным).*

Задача «Неинъективный рюкзак» («m-неинъективный рюкзак»).

Дано: *набор натуральных чисел $a_1, \ldots, a_n$.*

Определить: *является ли набор неинъективным (m-неинъективным).*

Очевидно, что задача **«2-инъективный рюкзак»** эквивалентна задаче **«Инъективный рюкзак»**, и задача **«m-инъективный рюкзак»** включает в себя задачу **«Инъективный рюкзак»**.

Ясно, что задачи **«Неинъективный рюкзак»** и **«m-неинъективный рюкзак»** являются дополнениями задач **«Инъективный рюкзак»** и **«m-инъективный рюкзак»** соответственно и принадлежат классу **NP**.

Следующие утверждения устанавливают к каким классам сложности относятся рассматриваемые задачи.

Теорема 3. *Задача* ***«Неинъективный рюкзак»*** *является* ***NP****-полной.*

Следствие 1. *Задача* ***«m-неинъективный рюкзак»*** *является* ***NP****-полной.*

Следствие 2. *Задачи* ***«Инъективный рюкзак»*** *и* ***«m-инъективный рюкзак»*** *являются* ***coNP****-полными.*

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О НЕКОТОРЫХ ЗАДАЧАХ ОПТИМАЛЬНОЙ ДОБЫЧИ ВОЗОБНОВЛЯЕМОГО РЕСУРСА

ON SOME PROBLEMS OF OPTIMAL HARVESTING OF RENEWABLE RESOURCE

Егорова А. В.[1], **Родина Л. И.**[2]

[1]*Владимирский государственный университет им. А.Г. и Н.Г. Столетовых, г. Владимир, Россия;*
nastik.e@bk.ru

[2]*Владимирский государственный университет им. А.Г. и Н.Г. Столетовых, г. Владимир, Россия;*
Lrodina67@mail.ru

Основным объектом исследования в данной работе являются модели динамики неоднородной популяции, заданные разностными уравнениями. Одна из таких моделей — модель двухвозрастной лимитированной популяции, подверженной промыслу — описана в работе [1]. Для данных моделей рассматриваются задачи оптимального сбора ресурса на бесконечном промежутке времени при различных ограничениях на условия промысла.

Обозначим через x_n и y_n количество ресурса каждого вида или возрастного класса в момент времени $n \in \{0, 1, 2, \ldots\}$. Пусть в момент n можно собирать доли ресурсов u_n, v_n каждого вида или одного из видов. Определим

$$U \doteq \{\bar{u} : \bar{u} = (u_0, u_1, \ldots, u_n, \ldots)\}, \;\; V \doteq \{\bar{v} : \bar{v} = (v_0, v_1, \ldots, v_n, \ldots)\}$$

и рассмотрим последовательности $\bar{u} \in U$, $\bar{v} \in V$ как управления, которыми можно варьировать для достижения лучшего результата сбора ресурса. Исследуем модель эксплуатируемой популяции

$$\begin{cases} x_{n+1} = f_1\big((1-u_n)x_n, (1-v_n)y_n\big), \\ y_{n+1} = f_2\big((1-u_n)x_n, (1-v_n)y_n\big), \end{cases} \qquad (1)$$

где $f_1(x, y)$, $f_2(x, y)$ — заданные для всех $(x, y) \in \mathbb{R}^2_+ \doteq \{(x, y) \in \mathbb{R}^2 : x \geqslant 0, y \geqslant 0\}$ вещественные непрерывные неотрицательные функции.

Пусть C_1, C_2 — стоимости условной единицы каждого из видов, $\alpha_1 \geqslant 0, \alpha_2 \geqslant 0$ — показатели дисконтирования, тогда стоимость всей добываемой продукции в момент n равна

$$h_n = C_1 x_n u_n e^{-\alpha_1 n} + C_2 y_n v_n e^{-\alpha_2 n}.$$

Для любых $(\bar{u}, \bar{v}) \in U \times V$ и $X_0 = (x_0, y_0) \in \mathbb{R}_+^2$ определим *среднюю временную выгоду* от извлечения ресурса при $\alpha_1 > 0, \alpha_2 > 0$ равенством

$$H_\alpha\big(\bar{u}, \bar{v}, X_0\big) \doteq \sum_{k=1}^{\infty} h_k$$

и при $\alpha_1 = \alpha_2 = 0$ равенством $H_0\big(\bar{u}, \bar{v}, X_0\big) \doteq \lim\limits_{n\to\infty} \dfrac{1}{n}\sum\limits_{k=0}^{n-1} h_k$.

Под *стационарным режимом эксплуатации* популяции будем понимать такой способ добычи ресурса, при котором $u_n = u \in [0,1]$, $v_n = v \in [0,1]$ для всех $n = 0, 1, 2, \ldots$ или для $n \geqslant N_0$. Допустим, что при данном режиме система (1) имеет устойчивую неподвижную точку $X^* = (x^*, y^*)$, где $x^* = x^*(u,v)$, $y^* = y^*(u,v)$. Пусть $D = D(X^*)$ является множеством притяжения точки X^*. Введем в рассмотрение множество $\widetilde{D} = \widetilde{D}(X^*) \doteq \bigcup\limits_{(x,y)\in D} [x, +\infty) \times [y, +\infty)$, функцию

$$F(x, y, u, v) = \big(f_1((1-u)x, (1-v)y), f_2((1-u)x, (1-v)y)\big)^*$$

и $F^m(x, y, u_0, \ldots, u_{m-1}, v_0, \ldots, v_{m-1})$ — m-ю итерацию функции F, $m = 1, 2, \ldots$. Пусть множество $F^{-m}(\widetilde{D})$ состоит из точек $(x, y) \in \mathbb{R}^2$ таких, что

$$F^m(x, y, u_0, \ldots, u_{m-1}, v_0, \ldots, v_{m-1}) \in \widetilde{D}$$

при некоторых $(u_0, \ldots, u_{m-1}, v_0, \ldots, v_{m-1}) \in [0,1]^{2m}$ и $F^0(\widetilde{D}) = \widetilde{D}$.

Теорема 1. *Предположим, что при стационарном режиме эксплуатации система (1) имеет устойчивую неподвижную точку $X^* = (x^*, y^*)$, где $x^* = x^*(u,v)$, $y^* = y^*(u,v)$. Тогда для любой начальной точки $X_0 \in \bigcup\limits_{m=0}^{\infty} F^{-m}(\widetilde{D})$ существуют управления $(\bar{u}, \bar{v}) \in U \times V$, такие, что выполнено равенство*

$$H_0\big(\bar{u}, \bar{v}, X_0\big) = C_1 x^*(u,v) u + C_2 y^*(u,v) v.$$

Рассмотрены примеры вычисления средней временной выгоды $H_0\big(\bar{u}, \bar{v}, X_0\big)$ для двухвозрастной популяции. При этом учитывается, что эксплуатируемая популяция обладает свойством мультирежимности, то есть при одних и тех же значениях параметров системы (1) наблюдаются различные динамические режимы. Например, при стационарном режиме изъятия только из младшей возрастной группы траектория, выпущенная из начальной точки, приходит либо в неподвижную точку, либо приближается к циклу длины три, в зависимости от начальных значений.

Для разностного уравнения первого порядка

$$x_{n+1} = f((1-u_n)x_n), \quad n = 0, 1, 2, \ldots$$

определим среднюю временную выгоду $H_\alpha\big(\bar{u}, x_0\big) \doteq \sum\limits_{k=0}^{\infty} x_k u_k e^{-\alpha k}$ при $\alpha > 0$ и $H_0\big(\bar{u}, x_0\big) \doteq \lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=0}^{n-1} x_k u_k$.

Теорема 2. *Пусть $G(x) \doteq f(x) - e^{\alpha} x$ достигает максимального значения при $\widetilde{x} > 0$ и для $x_0 > 0$ существует $k_0 = k_0(x_0)$ — наименьшее из чисел $\{0, 1, 2, \ldots\}$, такое, что $f^{k_0}(x_0) \geqslant \widetilde{x}$. Тогда наибольшее значение функций $H_\alpha\big(\bar{u}, x_0\big)$ и $H_0\big(\bar{u}, x_0\big)$ достигается при стационарном режиме эксплуатации:*

$$H_\alpha\big(\bar{u}^*, x_0\big) = \big(f^{k_0}(x_0) - \widetilde{x}\big)e^{-\alpha k_0} + \frac{f(\widetilde{x}) - \widetilde{x}}{1 - e^{-\alpha}}\, e^{-\alpha(k_0+1)},$$
$$H_0\big(\bar{u}^*, x_0\big) = f(\widetilde{x}) - \widetilde{x},$$

где $u_0^* = \ldots = u_{k_0-1}^* = 0$, $u_{k_0}^* = 1 - \dfrac{\widetilde{x}}{f^{k_0}(x_0)}$, $u_k^* = 1 - \dfrac{\widetilde{x}}{f(\widetilde{x})}$ для всех $k > k_0$.

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АНАЛИЗ УСТОЙЧИВОСТИ РЕШЕНИЙ НАЧАЛЬНО-КРАЕВОЙ ЗАДАЧИ ДЛЯ ПАРАБОЛИЧЕСКОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ОПЕРАТОРОМ ПОВОРОТА ПРОСТРАНСТВЕННОГО АРГУМЕНТА И ЗАПАЗДЫВАНИЕМ

Кубышкин Е. П.[1]**, Куликов В. А.**[2]

[1]*Ярославский государственный университет им.П.Г.Демидова, Ярославль, Россия;* kubysh.e@yandex.ru

[2]*Ярославский государственный университет им.П.Г.Демидова, Ярославль, Россия;* kulikov7677@gmail.com

В круге $K_R = \{(\rho,\phi) :\ 0 \le \rho \le R, 0 \le \phi < 2\pi\}$ рассматривается начально-краевая задача для параболического уравнения вида

$$u_t + u = D\Delta_{\rho\phi}u + bu_{\theta T},\quad u_\rho = 0 \tag{1}$$

относительно функции $u(\rho,\phi,t+s), t \ge 0, -T \le s \le 0$, где $T > 0$ величина запаздывания аргумента, с начальным условием $u(\rho,\phi,s) = u_0(\rho,\phi,s) \in H(K_R;-T,0)$– пространству начальных условий. В (1) $\Delta_{\rho\phi}$– оператор Лапласа в полярных координатах, $u_{\theta T}(\rho,\phi,t) \equiv u(\rho,\phi+\theta,t-T)(0 \le \theta < 2\pi)$ – оператор поворота пространственного аргумента и временного запаздывания, D, b – положительные постоянные.

Решается задача построения в пространстве параметров D, b, T, θ областей устойчивости решений начально-краевой задачи (1), исследуется характер потери устойчивости при прохождении параметров границы области устойчивости.

Определяя решения (1) вида $u(\rho,\phi,t) = v(\rho,\phi)e^{-\lambda t},\ \lambda \in \mathbb{C}$ получим пучек операторов

$$\lambda v(\rho,\phi) + v(\rho,\phi) - D\Delta_{\rho\phi}v(\rho,\phi) + bv(\rho,\phi+\theta)e^{-\lambda T},\ v_\rho(R,\phi) = 0, \tag{2}$$

спектр которого определяет устойчивость (неустойчивость) решений начально-краевой (1).

Определяя решение (2) в виде

$$v(\rho,\phi) = v_0 + \sum_{j=1}^{\infty}\sum_{n=-\infty}^{\infty} J_n(\gamma_{nj}\rho/R)(v_{nj}e^{in\phi} + v{-}nje^{-in\phi}),$$

($i = \sqrt{-1},\ v_0, v_{nj}, v_{-nj} \in \mathbb{C}$), где $J_n(\rho)$ функции Бесселя первого рода n - го порядка, а γ_{nj} j -й положительный ноль функции $J'_n(\rho)$, получим последовательность уравнений

$$(\lambda+1+D\gamma_{nj}^2+be^{in\theta-\lambda T})v_{nj}=0,\quad (\lambda+1+D\gamma_{nj}^2+be^{-in\theta-\lambda T})v_{-nj}=0, \tag{3}$$

из которых определяются точки спектра пучка операторов (2), отвечающие за устойчивость решений (1), и ненулевые v_0, v_{nj} $n = 0, \pm1, \pm2, ...$

Для построения границ областей устойчивости в пространстве параметров D, b, T, θ воспользуемся методом D - разбиений. Для этого положим в (3) $\lambda = i\omega,\ \omega > 0$ и приравняем нулю вещественную и мнимые части выражений в круглых скобках . Полученные уравнения позволяют выразить $b = b(D, T, n, \omega)$ и $\theta = \theta(D, T, n, \omega)$. Изменяя теперь ω, для различных значений D, T, n построим границы перехода точек спектра пучка операторов (2) из левой комплексной полуплоскости в правую и, тем самым, получим границу области устойчивости и исследуем характер потери устойчивости решений начально-краевой задачи (1).

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ОБ ОДНОМ ОСОБОМ СЛУЧАЕ ДЛЯ СЛАБОДИССИПАТИВНОГО КОМПЛЕКСНОГО УРАВНЕНИЯ ГИНЗБУРГА-ЛАНДАУ

ABOUT ONE SPATIAL CASE FOR THE WEAKLY DISSIPATIVE COMPLEX GINZBURG-LANDAU EQUATION

Куликов А. Н.[1], Куликов Д. А.[2]

[1]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* anat_kulikov@mail.ru

[2]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* kulikov_d_a@mail.ru

В работах [1-3] была рассмотрена периодическая краевая задача для слабодиссипативного варианта комплексного уравнения

Гинзбурга-Ландау в случае произвольного числа пространственных переменных

$$u_t = u - (1 + ic)u|u|^2 - id\Delta u, \qquad (1)$$

где $u = u(t, x_1, \ldots, x_n), n \in N, d, c \in R, d > 0, \Delta u$ – оператор Лапласа по пространственным переменным. В этих работах были указаны автомодельные периодические по t решения вида $\exp(i\sigma_m t + i(m, x))$,

где $\sigma_m = -c + d \sum_{j=1}^{n} m_j^2$,

$(m, x) = \sum_{j=1}^{n} m_j x_j, m_j \in Z$. Был исследован вопрос об их устойчивости и локальных бифуркациях инвариантных торов.

В докладе ограничимся рассмотрением частного случая, когда $n = 1$ и рассмотрим краевую задачу

$$u_t = u - (1 + ic)u|u|^2 - idu_{xx}, \qquad (2)$$

$$u(t, x + 2\pi) = u(t, x). \qquad (3)$$

При этом рассмотрим особый случай и будем считать, что $\varepsilon = \dfrac{1}{d} << 1$. После замены $s = td$ краевую задачу (2), (3) можно записать в виде

$$u_s = -iu_{xx} + \varepsilon u[1 - (1 + ic)|u|^2], \qquad (4)$$

$$u(s, x + 2\pi) = u(s, x). \qquad (5)$$

Теорема. *Существует такая положительная постоянная $\varepsilon_0 = \varepsilon_0(n)$, что при всех $\varepsilon \in (0, \varepsilon_0)$ краевая задача (4), (5) имеет торы $T_p(\varepsilon)$ любой размерности $p \leq n$. Одномерные "торы" (циклы) устойчивы, а торы иной размерности,отличной от 1, седловые.*

Для решений, принадлежащих инвариантным торам $T_p(\varepsilon)$ $(dimT_p(\varepsilon) = p)$ справедливы асимптотические формулы

$$u(t, x, \varepsilon) = \frac{2}{\sqrt{2p - 1}} \sum_{k=1}^{p} \cos((n_k^2 - \varepsilon c)t + n_k x + \varphi_{n_k}) + O(\varepsilon),$$

где $\{n_k\}$ – набор из p целых чисел, а $\varphi_{n_k} \in R$ и произвольны, а $p \leq n$.

Обоснование теоремы основано на использовании аппарата теории нормальных форм и утверждений о сохранении инвариантных торов при возмущениях.

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ДИНАМИЧЕСКИЕ СИСТЕМЫ С КОЛЕБАТЕЛЬНО УБЫВАЮЩИМИ КОЭФФИЦИЕНТАМИ: НЕКОТОРЫЕ ЗАДАЧИ ТЕОРИИ АСИМПТОТИЧЕСКОГО ИНТЕГРИРОВАНИЯ

DYNAMICAL SYSTEMS WITH OSCILLATORY DECREASING COEFFICIENTS: CERTAIN PROBLEMS OF ASYMPTOTIC INTEGRATION THEORY

Нестеров П. Н.

ЯрГУ им П.Г. Демидова, г. Ярославль, Россия;
p.nesterov@uniyar.ac.ru

В докладе излагаются некоторые общие результаты, лежащие в основе задачи получения асимптотических представлений для решений линейных динамических систем с так называемыми колебательно убывающими коэффициентами при стремлении независимой переменной к бесконечности. Речь здесь идет как о конечномерных линейных системах ОДУ

$$\dot{x} = \big[A + G(t)\big]x, \qquad x \in \mathbb{C}^m, \tag{1}$$

так и о бесконечномерных системах функционально-дифференциальных уравнений

$$\dot{x} = Ax_t + G(t, x_t), \qquad x \in \mathbb{C}^m, \quad x_t(\theta) = x(t+\theta), \quad -h \le \theta \le 0, \tag{2}$$

а также об уравнениях в банаховом пространстве

$$\dot{x} = \big[A + G(t)\big]x, \qquad x \in \mathcal{B}. \tag{3}$$

В этих уравнениях матрица или оператор A не зависит от переменной t, а матрица или операторная функция $G(\cdot)$ является некоторым возмущением, которое, в свою очередь, от переменной t уже зависит. Причем возмущение $G(\cdot)$ является в определенном смысле малым. Именно, оно состоит из двух компонент, одна из которых является абсолютно интегрируемой на промежутке $[t_0, \infty)$ матрицей или операторной функцией, а другая — стремится к нулю колебательным образом при $t \to \infty$. Таким образом, рассматриваемые уравнения являются определенного рода возмущениями систем с постоянными коэффициентами. Как оказывается, вид колебательно убывающей составляющей возмущения $G(\cdot)$ может существенным образом влиять на динамику решений указанных уравнений.

Мы приведем несколько результатов, которые позволяют строить асимптотики для решений уравнений (1) — (3) при $t \to \infty$. В частности, мы поговорим о классической теореме Н. Левинсона [1], обсудим идею усредняющих замен переменных [2], а также воспользуемся идеологией теории центральных многообразий [3] для решения задачи асимптотического интегрирования [4,5]. Мы проиллюстрируем предложенную методику на примере построения асимптотических формул для решений конкретных динамических систем.

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БЕЗОПАСНОСТЬ ПОТОКОВ ДАННЫХ В ПРОГРАММНО-КОНФИГУРИРУЕМЫХ СЕТЯХ

INFORMATION FLOW SECURITY IN SOFTWARE-DEFINED NETWORKS

Соколов В. А.[1], Чалый Д. Ю.[2]

[1]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* sokolov@uniyar.ac.ru
[2]*Ярославский государственный университет им. П.Г. Демидова, Ярославль, Россия;* chaly@uniyar.ac.ru

Программно-конфигурируемые сети являются современной, более гибкой парадигмой построения программно-конфигурируемых сетей, которая позволяет программным образом управлять сетевыми ресурсами. Для этого в сети выделяется управляющая сущность — контроллер, на котором выполняется программный код, устанавливающий правила передачи трафика на управляемых коммутаторах. При этом может быть повышена безопасность коммуникационной сети, поскольку программно-конфигурируемые сети предоставляют гораздо более богатый набор средств для управления сетью, чем традиционные подходы, основанные на конфигурировании и мониторинге устройств.

Обеспечение безопасности передачи данных в программно-конфигурируемых сетях является важной и актуальной задачей. При этом необходимо обеспечить безопасность каналов передачи данных от контроллера к управляемым устройствам. В том случае, когда контроллер является распределенным, необходимо обеспечить безопасность между управляющими сущностями контроллера. Важным является верификация программного обеспечения, которое реализует логику управления, поскольку наличие ошибок, ведущих к сбоям, может существенно снижать качество сервиса и создавать угрозы безопасности коммуникационной сети. Еще одной задачей является разработка средств, способных осуществлять мониторинг соответствия постоянно изменяющейся конфигурации программно-конфигурируемой сети тем требованиям безопасности, которые к ней предъявляются.

Одним из таких подходов является анализ информационных потоков, т.е. анализ того, как коммуникационный трафик может распространяться по сети. При этом основной задачей является недопущение передачи трафика от сущностей с повышенным уровнем безопасности к тем, которые имеют более низкий уровень. Такими сущностями могут быть как отдельные узлы сети, так и отдельные приложения. При этом во втором случае необходимо учитывать, что на одном вычислительном узле могут исполняться разные категории приложений. Тогда ответственность за исключение нежелательных информационных потоков переносится на уровень операционной системы.

В данной концепции предполагается, что злоумышленник может иметь доступ к узлам сети, которые имеют более низкий уровень безопасности, либо к каналам связи. Следовательно, необходимо учитывать, что секретные данные могут быть перехвачены или искажены злоумышленником. При этом изоляция потоков, которая представляется очевидным решением, не позволяет решать ряд задач, например, мониторинг узлов с низким уровнем безопасности узлами с высоким уровнем безопасности.

Все это приводит нас к поиску решений поставленной задачи с помощью логических средств. Поскольку стандарт программно-конфигурируемых сетей определяет ограниченный набор управляющих примитивов, каждый из которых имеет достаточно точное определение, мы для каждого из них можем определить правила логического вывода, которые позволяют определить, допускает ли такой примитив нарушения свойств безопасности или нет. Полученная система логического вывода дополняется правилами, которые позволяют анализировать также конкатенацию управляющих команд. В результате мы получаем логическую систему, которая позволяет на лету анализировать состояние сети и сигнализировать о возникающих ошибках конфигурации, которые могут приводить к явным или неявным утечкам конфиденциальных данных.

Можно рассмотреть следующие перспективные направления исследований. Как мы уже указали ранее, изоляция информационных потоков является решением, которое не допускает многих полезных приложений. В связи с этим можно предложить подходы, для которых логическая система последовательно строится исходя из требований безопасности и модели злоумышленника. Перспективным также является изучение методов деклассификации, т.е. такого способа понижения уровня секретности данных, которое не нарушает требования безопасности. Можно также предлагать логические системы, которые идентифицируют нарушение одного из классов свойств безопасности, например, целостности или конфиденциальности.

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