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Singularities of integrable systems: a criterion for non-degeneracy and a generalization monodromy, with an application to the Manakov top

Тонконог Дмитрий Иванович

Студент

Московский государственный университет имени М.В. Ломоносова,

Механико-математический факультет, Москва, Россия

E-mail: dtonkonog@gmail.com

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A (completely) integrable Hamiltonian system $(M, \omega, h_1, \dots, h_n)$ is a symplectic $2n$ -manifold (M, ω) with functionally independent commuting functions $h_1, \dots, h_n : M \rightarrow \mathbb{R}$ called *integrals*. The *momentum map* $\mathcal{F} : M \rightarrow \mathbb{R}^n$ given by $\mathcal{F}(x) := (h_1(x), \dots, h_n(x))$. The Hamiltonian vector field of a function g on M is denoted by $\text{sgrad } g$. A point $x \in M$ is called a *singular (critical) point of rank r* , $0 \leq r$, if $\text{d}\mathcal{F}|_x = r$. For such points, there is a natural notion of non-degeneracy [1], [2]. We recall this definition for zero-rank critical points below.

In general, non-degenerate singularities are important because they are generic and because the structure of integrable systems in their neighborhood is well understood, see survey [3]. We present a geometric criterion for non-degeneracy of a singularities of integrable Hamiltonian systems, see Theorem 1 below.

The criterion is applied to prove non-degeneracy of the saddle-saddle singularity in the Manakov top system [4]. We use the proved non-degeneracy and Fomenko theory [2] to obtain explicit semilocal description of the singularity. This description allows to detect a phenomenon which appears in the Manakov top. It naturally generalizes Hamiltonian monodromy introduced by Duistermaat [5] and bidromy introduced recently by Sadovskii and Zhilinskii [6]. We call it *partial monodromy*. Sinitsyn and Zhilinskii [4] showed that no Hamiltonian monodromy in any previously known sense appears in the Manakov top and encouraged further study of the system; our results develop the topic.

Definition 1. Let $(M, \omega, h_1, \dots, h_n)$ be an integrable Hamiltonian system. A zero-rank singular point $P \in M$ is called *non-degenerate* if the commutative subalgebra K of $\text{sp}(2n, \mathbb{R})$ generated by linear parts of Hamiltonian vector fields $\text{sgrad } h_1, \dots, \text{sgrad } h_n$ at point P is a Cartan subalgebra of $\text{sp}(2n, \mathbb{R})$.

Theorem 1. Consider a completely integrable Hamiltonian system $(M, \omega, h_1, \dots, h_n)$. Let $\mathcal{F} : M \rightarrow \mathbb{R}^n$ be the momentum map and $P \in M$ be a zero-rank singular point of the system. Denote by K the set of all singular points of rank 1 in a neighborhood of P . If the following conditions hold, then P is non-degenerate:

- (a) There exists a non-degenerate linear combination of forms $\{\text{d}^2 h_i|_P\}_{i=1}^n$.
- (b) The image $\mathcal{F}(K)$ contains n smooth curves $\gamma_1, \dots, \gamma_n$, each curve having P as its end point or its inner point (examples for $n = 2$ are found on the figure). The vectors tangent to $\gamma_1, \dots, \gamma_n$ at $\mathcal{F}(P)$ are independent in \mathbb{R}^n .
- (c) K is a smooth submanifold of M or, at least, $K \cup \{P\}$ coincides with the closure of the set of all points $x \in K$ having a neighborhood $V(x) \subset M$ for which $K \cap V(x)$ is a smooth submanifold of M .

Remark. Notice that condition (c) is very weak. For example, it automatically holds

if the integrals h_i are polynomials (in a suitable system of local coordinates at point P) because in this case each D_i is given by a system of algebraic equations.

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Иллюстрации

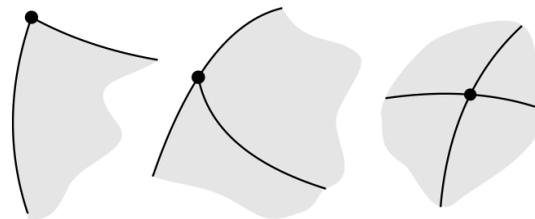


Рис. 1: Bifurcation diagrams satisfying condition (b) of Theorem 1.