Секция «Математика и механика»

Little's Theorem for queueing systems with regenerative inflow.

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This presentation is about a study of queueing systems with regenerative inflow. In particular modification of Little's Theorem would be stated and proved.

Definition 1 *Point process $N(t)$ is called regenerative flow, if following conditions are at place:*

i) Exists an ascending sequence of Markov times $\{\tau_k\}_{k=0}^{\infty}, \tau_k < \tau_{k+1}$ in a relation to a certain filtration $\{\mathcal{F}_{\leq t}\}$, which forms a renewal process, i.e. $\{\tau_k - \tau_{k-1}\}_{k=1}^{\infty}$ are independent identically distributed random variables. Variable τ_k is called $k-th$ regeneration moment of the flow and $\theta_k = \tau_k - \tau_{k-1}$ is a $k-th$ period of regeneration.

ii) Sequence $\{\theta_k, N(t) - N(\tau_{k-1}), t \in [\tau_{k-1}, \tau_k)\}_{k=1}^{\infty}$ forms a sequence of independent identically distributed random elements.

1) From the definition of $N(t)$ follows that $\{\theta_k, \xi_k\}_{k=1}^{\infty}$ are independent identically distributed random vectors, where $\xi_k = N(\tau_k) - N(\tau_{k-1})$ is a number of demands arrived into system during k-th period of flow's $N(t)$ regeneration.

2) r_k is an in-service time for k-th demand and r_k are form a sequence of independent identically distributed random variables.

3) $q(t)$ is a number of demands in the system at the moment t (sum of demands in the queue and in service). Also existence of mean values $E\xi_k$, Er_k, $E\theta_k$ is assumed.

Let $W(t)$ a stochastic process of virtual waiting time (or workload process), i.e. required time for the system to get free on the condition that new demands are not arriving, the same as workload of the system at moment t (assuming servicing speed is equal to 1). Consider imbedded process $W_n = W(\tau_n - 0)$ (i.e. workload of the system at moments τ_n - $N(t)$'s regenerations). Also consider auxiliary processes W_n^- and W_n^+, processes in which all the work from n-th period arrives at the start and at the end of the period respectively.

Theorem 1 *We prove following inequations:*

$$W_n^- \overset{D}{\leq} W_n \overset{D}{\leq} W_n^+.$$

From the stochastic boundedness of W_n^- and W_n^+ follows stochastic limitation of W_n.

Theorem 2 *Process W_n is stochastically bounded if and only if following condition is at place:*

$$\rho = \frac{E\xi_k}{E\theta_k} \cdot Er_k < 1$$

ρ is called a usage coefficient.

Consequently W_n is a regenerative process with $\{n : W_n = 0\}$ as

regeneration moments. Using results of Smith theorem from stochastic boundedness of regenerative process W_n follows existence of mean value for regeneration period of W_n. Thus mean value for regeneration period of process $\{W(t), q(t)\}$ also exists, where $T_n = \inf_k \{\tau_k : W(\tau_k) = 0, \tau_k > T_{n-1}\}$ are respective regeneration moments.

Suggestion 1 *If following conditions are at place:*

1) distribution of θ_k has absolutely continuous component,

2) usage coefficient ρ<1,

3) probability of event that nothing arrived during $k-th$ regeneration period of $N(t)$ is greater than 0,

Then there is limit distribution for number of demands in the system

$$q(t) \xrightarrow{D} \tilde{q} \qquad \textit{when } t \to \infty.$$

Now we prove following

Theorem 3 *Sequence $\{\{a_{n+k}\}_{k=1}^{\infty}\}_{n=0}^{\infty}$ when $n \to \infty$ weakly converge to stationary sequence i.e. for any m there is following distribution convergence:*

$$(a_{n+1}, ..., a_{n+m}) \xrightarrow{D} (\tilde{a}_1, ..., \tilde{a}_m) \qquad \textit{when } n \to \infty$$

Theorem 4 *Sequence $\{\{v_{n+k}\}_{k=1}^{\infty}\}_{n=0}^{\infty}$ when $n \to \infty$ weakly converge to stationary sequence i.e. for any m there is following distribution convergence:*

$$(v_{n+1}, ..., v_{n+m}) \xrightarrow{D} (\tilde{v}_1, ..., \tilde{v}_m) \qquad \textit{when } n \to \infty$$

Theorem 5 *Sequence of stochastic processes $\{q(t+T)\}_{T \in \mathbb{R}^+}$ when $T \to \infty$ weakly converge to a stationary process i.e. for any set $t_1, ..., t_m$ there is following distribution convergence:*

$$(q(t_1 + T), ..., q(t_m + T)) \xrightarrow{D} (\tilde{q}(t_1), ..., \tilde{q}(t_m)) \qquad \textit{when } T \to \infty$$

Name *inflow intensity* variable $\lambda = \frac{E\xi_k}{E\theta_k}$. Now we prove following:

Suggestion 2

$$\frac{N(t)}{t} \xrightarrow{a.s.} \lambda \qquad \textit{when } t \to \infty.$$

We call the process $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{v}_1, \tilde{v}_2, ..., \tilde{q}(t))$ *a stationary version* of regenerative process $(a_1, a_2, ..., v_1, v_2, ..., q(t))$. Common Little's Theorem is applicable to process $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{v}_1, \tilde{v}_2, ..., \tilde{q}(t))$ Now we formulate Little's Theorem for queueing systems with regenerative inflow:

Theorem 6 ***(Little's Theorem for queueing systems with regenerative inflow)***

If L, V, λ are mean limit value for number of demands in the system, mean limit value of sojourn times, intensity of inflow respectively, then following condition takes place:

$$L = \lambda V.$$