Local cohomology functors for Noetherian Grothendieck categories

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Let R be a commutative Noetherian ring, I an ideal of R, and M an R-module. Set

$$\Gamma_I(M) = \{x \in M | \textit{there exists } n \in \mathbb{N} \textit{ such that } I^n x = 0\},$$

and let $\mathbf{R}\Gamma_I : \mathbf{D}^+(Mod(R)) \to \mathbf{D}^+(Mod(R))$ be the right derived functor of $\Gamma_I : Mod(R) \to Mod(R)$. The section functor Γ_I and the local cohomology functor $\mathbf{R}\Gamma_I$ are widely used to Commutative algebra and Algebraic geometry. They are actually extensively studied by many authors. As for the section functor Γ_I, Yuji Yoshino and Takeshi Yoshizawa in their article consider the set of all the left exact radical functors on $Mod(R)$. Actually, Γ_I is a left exact radical functor:

Теорема 1. *The following conditions are equivalent for a left exact preradical functor γ on $Mod(R)$.*

- *(1) γ is a radical functor.*
- *(2) γ preserves injectivity.*
- *(3) γ is a section functor with support in a specialization-closed subset of $Spec(R)$.*
- *(4) $\mathbf{R}\gamma$ is an abstract local cohomology functor.*

Our strategy is the following. Let X be a Noetherian Grothendieck category. (A Grothendieck category is an Abelian category which has coproducts, in which direct limits are exact and which has a generator. An object M in the category X is Noetherian if each ascending chain of subobjects of M is stationary. A Grothendieck category X will be called Noetherian if X has a Noetherian generator.) Let V be a localizing subset of $ASpec(X)$. We can define the section functor Γ_V with support in V as

$$\Gamma_V(M) = \bigcup \{N \subset M | ASupp(N) \subset V\} \tag{1}$$

for all $M \in X$. A functor $F : X \to X$ is called a preradical functor if F is a subfunctor of $\mathbf{1}$. A preradical functor F is called a radical functor if $F(M/F(M)) = 0$ for all $M \in Obj(X)$. We denote $\mathcal{C} = \mathbf{D}^+(X)$. Let $\delta : \mathcal{C} \to \mathcal{C}$ be a triangle functor. We call that δ is an abstract local cohomology functor if the following conditions are satisfied:

- (1) The natural embedding functor $i : Im(\delta) \to \mathcal{C}$ has a right adjoint $\rho : \mathcal{C} \to Im(\delta)$ and $\delta \cong i \circ \rho$.
- (2) The t-structure $(Im(\delta), Ker(\delta))$ divides indecomposable injective objects.

We will show that a left exact radical functor F is of the form Γ_V for a localizing subset V:

Теорема 2. *The following conditions are equivalent for a left exact preradical functor F on X.*

- *(1) F is a radical functor.*
- *(2) F preserves injectivity.*
- *(3) F is a section functor with support in a localizing subset of $ASpec(X)$.*
- *(4) $\mathbf{R}F$ is an abstract local cohomology functor.*