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Секция «Математическая логика, алгебра и теория чисел»

The chromatic number of space with Chebyshev metric without long monochromatic unit arithmetic progressions

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Аспирант

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ProblemProblem

Given a natural n, we construct a two-coloring of an n-dimensional space with the maximum metric satisfying the following: for any finite set of reals S with diameter greater than 5^n such that the distance between any two consecutive points of S does not exceed one, no isometric copy of S is monochromatic. As a corollary, we prove that any normed space can be two-colored such that all sufficiently long unit arithmetic progressions contain points of both colors.

For an n-dimensional normed space $\mathbb{R}^n_N$, the *chromatic number* $\chi(\mathbb{R}^n_N)$ is the smallest r such that there exists a coloring of the points of $\mathbb{R}^n_N$ with r colors, i.e. an *r-coloring*, and with no two points of the same color unit distance apart. A general result by Kupavskii [3] establishes an upper bound on this quantity depending only on the dimension: $\chi(\mathbb{R}^n_N) \le (4+o(1))^n$ as $n \to \infty$.

Given a normed space $\mathbb{R}^n_N$ and a subset $\mathcal{M} \subset \mathbb{R}^n$, the *chromatic number* $\chi(\mathbb{R}^n_N, \mathcal{M})$ is the smallest r such that there exists an r-coloring of $\mathbb{R}^n$ with no monochromatic *N-isometric copy*[1] of $\mathcal{M}$. In these terms, $\chi(\mathbb{R}^n_N) = \chi(\mathbb{R}^n_N, I)$, where I is a two-point set.

We consider a sequence of positive reals $\lambda_1, \dots, \lambda_k$, given in [4]. We call a set $\{0, \lambda_1, \lambda_1 + \lambda_2, \dots, \sum_{t=1}^k \lambda_t\} \subset \mathbb{R}$ a *baton* and denote it by $\mathcal{B}(\lambda_1, \dots, \lambda_k)$. In case $\lambda_1 = \dots = \lambda_k = 1$, i.e., if the set is just a unit arithmetic progression, we simply denote it by $\mathcal{B}_k$ for a shorthand. Erdős et al. [5] showed that any baton $\mathcal{B}$ of at least three points is not Ramsey, since $\chi(\mathbb{R}^n_2, \mathcal{B}) \le 16$ for all $n \in \mathbb{N}$. Moreover, they proved that

$$\chi(\mathbb{R}^n_2, \mathcal{B}_k) = 2 \tag{1}$$

for all $k \ge 5$ and for all natural n. (Note that it is unknown [1] whether the values 5 and 16 here are tight.)

Is it true that for any normed space $\mathbb{R}^n_N$, there is $k = k(\mathbb{R}^n_N)$ such that $\chi(\mathbb{R}^n_N, \mathcal{B}_k) = 2$?

In a recent series of papers [4, 6, 7], the authors studied the chromatic numbers $\chi(\mathbb{R}^n_\infty, \mathcal{M})$ for the n-dimensional Chebyshev spaces $\mathbb{R}^n_\infty$. Among the other results, they proved that

$$\chi(\mathbb{R}^n_\infty, \mathcal{B}_k) \ge \left(\frac{k+1}{k}\right)^n \tag{2}$$

for all $k, n \in \mathbb{N}$. This inequality shows that, unlike the Euclidean case, for any given k, every two-coloring of $\mathbb{R}^n$ contains a monochromatic ℓ_∞-isometric copy of $\mathcal{B}_k$ whenever the dimension n is large enough in terms of k. The next theorem shows that this is not that case in the 'opposite' setting, when k is sufficiently large in terms of n.

[1] A subset $\mathcal{M}' \subset \mathbb{R}^n$ is called an *N-isometric copy* of $\mathcal{M}$ if there exists a bijection $f : \mathcal{M} \to \mathcal{M}'$ such that $\|\mathbf{x} - \mathbf{y}\|_N = \|f(\mathbf{x}) - f(\mathbf{y})\|_N$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{M}$.

Теорема . *$th_m ainGivenn \in \mathbb{N}$, there exists a two-coloring of $\mathbb{R}^n$ with no monochromatic ℓ_∞-isometric copies of all batons $\mathcal{B}(\lambda_1, \ldots, \lambda_k)$ such that $\max_t \lambda_t \leq 1$ and $\sum_{t=1}^k \lambda_t \geq 5^n$. In particular, for all $n \in \mathbb{N}$ and $k \geq 5^n$, we have $\chi(\mathbb{R}^n_\infty, \mathcal{B}_k) = 2$.*

For the prove of the theorem we introduced the next notation: a *snake hypersurface* $\mathfrak{S}^n(a_1, b_1, \ldots, a_n$ where $n \in \mathbb{N}$, is an n-dimensional piecewise linear hypersurface in $\mathbb{R}^{n+1}$ that depends on $2n$ positive real parameters. The definition is by induction on n. Let $\mathfrak{S}^0(\varnothing)\{0\}$ be just the origin of the line. For $n > 0$, we define $\mathfrak{S}^n(a_1, b_1, \ldots, a_n, b_n)$ by

$$\mathfrak{S}^n(a_1, b_1, \ldots, a_n, b_n) = \mathfrak{S}^{n-1}(a_1, b_1, \ldots, a_{n-1}, b_{n-1}) + \mathbb{Z} \cdot \{a_n \cdot \mathbf{e}_{n+1} - b_n \cdot \mathbf{1}_n\} + [0, a_n) \cdot \mathbf{e}_{n+1} \cup (0, b_n] \cdot \mathbf{1}_n, \quad (3)$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors for $\mathbb{R}^n$, and $\mathbf{1}_n \in \mathbb{R}^n$ be their sum, i.e., a vector with all n coordinates being unit[1].

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[1]Note that here and in what follows we do not distinguish points from their position vectors.