2023

P.G. Demidov Yaroslavl State University
Ярославский государственный университет им. П.Г. Демидова

4th International Conference on Integrable Systems & Nonlinear Dynamics

2023

INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS (ISND - 2023)

Yaroslavl, September 25–29, 2023

CIS Centre of Integrable Systems

ТЕНЗОР

P. G. Demidov Yaroslavl State University
Regional Scientific and Educational Mathematical Center
"Centre of Integrable Systems"
"Tensor" IT company

INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS (ISND–2023)

Abstracts

4th International Conference on
INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS
(ISND–2023)

Yaroslavl, September 25–29, 2023

Ярославский государственный университет
им. П. Г. Демидова

Региональный научно-образовательный
математический центр
“Центр интегрируемых систем”

IT компания “Тензор”

ИНТЕГРИРУЕМЫЕ СИСТЕМЫ И НЕЛИНЕЙНАЯ ДИНАМИКА (ISND–2023)

Тезисы докладов конференции

4-я Международная конференция
"ИНТЕГРИРУЕМЫЕ СИСТЕМЫ
И НЕЛИНЕЙНАЯ ДИНАМИКА"
(ISND–2023)

Ярославль, 25—29 сентября 2023 г.

INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS
Yaroslavl, September 25–29, 2023

ORGANIZERS

P.G. Demidov Yaroslavl State University
Regional Scientific and Educational Mathematical Center
"Centre of Integrable Systems"
"Tensor" IT company

PROGRAM COMMITTEE

- Valery V. KOZLOV, Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow, Russia (chairman);
- Anastasios (Tassos) BOUNTIS, P.G. Demidov Yaroslavl State University (co-chairman);
- Jinsong HE, Shenzhen University, China (co-chairman);
- Sergey A. KASHCHENKO, P.G. Demidov Yaroslavl State University, Russia (co-chairman);
- Victor M. BUCHSTABER, Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow, Russia;
- Sergey D. GLYZIN, P.G. Demidov Yaroslavl State University, Russia;
- Ilya S. KASHCHENKO, P.G. Demidov Yaroslavl State University, Russia;
- Sotiris KONSTANTINOU-RIZOS, P.G. Demidov Yaroslavl State University, Russia;
- Vladimir V. SOKOLOV, Kharkevich Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow, Russia;
- Dmitry V. TALALAEV, Moscow State University & P.G. Demidov Yaroslavl State University, Russia;
- Dmitry V. TRESCHEV, Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow, Russia;
- Pavlos XENITIDIS, Liverpool Hope University, UK.

ORGANIZING COMMITTEE

S. G. KONSTANTINOU-RIZOS (chairman), P. N. NESTEROV (co-chairman), A. O. TOLBEY (secretary), S. A. KASHCHENKO, S. D. GLYZIN, S. A. IGONIN

UDC 517.95+517.938

INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS (ISND-2023) : Abstracts. – Yaroslavl: Filigran, 2023 . – 156 p. – (4th International Conference on INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS (ISND–2023), September 25–29, 2023, Yaroslavl).

ISBN 978-5-6050859-1-1

The P. G. Demidov Yaroslavl State University is hosting an 4th International Conference on INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS from 25 to 29 September 2023 in the city of Yaroslavl. This is a collection of abstracts of the conference talks. The abstracts published here are edited by the authors. Abstracts are printed in English.

The conference is held within framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher Education of the Russian Federation (Agreement on provision of subsidy from the federal budget No. 075-02-2023-948).

UDC 517.95+517.938

ISBN 978-5-6050859-1-1

CONTENTS

CONTENTS

NEW EXAMPLES OF INTEGRABLE MAGNETIC GEODESIC FLOWS ON 2-SURFACES

S. V. Agapov

Sobolev Institute of Mathematics, Novosibirsk, Russia;
Novosibirsk State University, Novosibirsk, Russia;
agapov.sergey.v@gmail.com

In this talk we will consider integrable magnetic geodesic flows on 2-surfaces. We will describe various methods which allowed us to construct numerous new integrable examples of such flows with polynomial and non-polynomial first integrals at a fixed energy level ([1], [2]). Also some results on polynomial integrals at several energy levels will be presented ([3], [4]).

REFERENCES

1. *Agapov S. V., Bialy M., Mironov A. E.,* “Integrable magnetic geodesic flows on 2-torus: new examples via quasi-linear system of PDEs,” Comm. Math. Phys., **351**, No. 3, 993–1007 (2017).
2. *Agapov S., Potashnikov A., Shubin V.,* “Integrable magnetic geodesic flows on 2-surfaces,” Nonlinearity, **36**, No. 4, 2128–2147 (2023).
3. *Agapov S., Valyuzhenich A.,* “Polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels,” Disc. Cont. Dyn. Syst. - A, **39**, No. 11, 6565–6583 (2019).
4. *Agapov S., Valyuzhenich A. A., Shubin V. V.,* “Some remarks on high degree polynomial integrals of the magnetic geodesic flow on the two-dimensional torus,” Siberian Math. J., **62**, No. 4, 581–585 (2021).

NIJENHUIS OPERATORS AND PROJECTIVELY EQUIVALENT METRICS IN SMALL DIMENSIONAL CASES

D. Zh. Akpan[1,2]

[1] *Lomonosov Moscow State University, Moscow, Russia;* akpandz@my.msu.ru
[2] *Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia;* akpandz@my.msu.ru

The talk is devoted to the study of Nijenhuis operators of arbitrary dimension n in a neighborhood of a point at which the first $n-1$ coefficients of the characteristic polynomial are functionally independent, and the last coefficient (the determinant of the operator) is an arbitrary function. Also we will see some examples of Nijenhuis operators in small-dimensional cases.

Definition 1. *Let M^n be a smooth n-dimensional manifold and let L be a tensor field of type $(1,1)$. Then the Nijenhuis torsion or the Nijenhuis tensor N_L is a tensor of type $(1,2)$ which is invariantly defined as follows:*

$$N_L[u,v] = L^2[u,v] + [Lu, Lv] - L[u, Lv] - L[Lu, v],$$

where u, v are arbitrary vector fields and $[u,v]$ is their commutator. In coordinates, this condition is written as follows:

$$(N_L)^i_{jk} = L^l_j \frac{\partial L^i_k}{\partial x^l} - L^l_k \frac{\partial L^i_j}{\partial x^l} - L^i_l \frac{\partial L^l_k}{\partial x^j} + L^i_l \frac{\partial L^l_j}{\partial x^k}.$$

Definition 2. *A tensor field L of type $(1,1)$, i.e., an operator field, is called a Nijenhuis operator if its Nijenhuis torsion is identically zero, i.e., $N_L \equiv 0$.*

Let L be a Nijenhuis operator, and let $\chi(t) = \det(t \cdot \mathrm{Id} - L) = t^n + \sigma_1 t^{n-1} + \ldots + \sigma_n$ be its characteristic polynomial. In [1], a theorem is proved on the form of the matrix of the Nijenhuis operator with functionally independent coefficients of the characteristic polynomial (invariants):

$$L = J^{-1} \tilde{L} J,$$

where

$$\tilde{L} = \begin{bmatrix} -\sigma_1 & 1 & 0 & \dots & 0 \\ -\sigma_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\sigma_n & 0 & 0 & \dots & 0 \end{bmatrix}, \qquad J = \Big(\frac{\partial \sigma_i(x)}{\partial x^j}\Big).$$

For example, in the two-dimensional case, if the trace $trL = x$ is the first coordinate, and $\det L = f(x,y)$ is a smooth function, then $\sigma_1 = -x$, $\sigma_2 = f(x,y)$ and we obtain the general form of these operators:

$$L = \begin{bmatrix} -1 & 0 \\ \frac{f_x}{f_y} & \frac{1}{f_y} \end{bmatrix} \begin{bmatrix} x & 1 \\ -f & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ f_x & f_y \end{bmatrix} = \begin{bmatrix} x - f_x & -f_y \\ \frac{-xf_x + f_x^2 + f}{f_y} & f_x \end{bmatrix},$$

where f_x, f_y stand for the corresponding partial derivatives of the function f. It is clear that the existence of a smooth two-dimensional Nijenhuis operator in the situation under consideration is equivalent to the smoothness of the fraction $\frac{xf_x - f_x^2 - f}{f_y}$. At the points at which $f_y \neq 0$, this fraction is obviously smooth. A more interesting case occurs when $f_y = 0$ at some point, but $\frac{xf_x - f_x^2 - f}{f_y}$ is smooth in its neighborhood. The Morse and cubic singularities of the determinant (restricted to the level line of the trace) in the two-dimensional case were investigated in [6].

In two-dimensional case we have 4 Nijenhuis operators with non-degenerate trace and with the Morse singularity of the determinant.

$$L_1^{\pm} = \begin{bmatrix} x & \pm 2\tilde{y} \\ \frac{\tilde{y}}{2} & 0 \end{bmatrix} \quad \text{and} \quad L_2^{\pm} = \begin{bmatrix} \frac{x}{2} & \pm 2\tilde{y} \\ \frac{\tilde{y}}{2} & \frac{x}{2} \end{bmatrix}.$$

As we can see, the determinant is equal to $f(x,y) = \pm y^2$ or $f(x,y) = \pm y^2 + \frac{x^2}{4}$.

It's an interesting effect that we got two solutions. In the multidimensional case this effect is not observed.

Two (pseudo)-Riemannian metrics g and $\bar{g}$ (of any, possibly different, signatures) are called geodesically equivalent if they share the same geodesics viewed as unparameterized curves. According to [5], a manifold endowed with a pair of such metrics carries a natural Nijenhuis structure defined by the operator

$$\nabla_k R_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

where $R = |\frac{det\bar{g}}{detg}|^{\frac{1}{n+1}} g\bar{g}^{-1}g$, $\lambda_i = \frac{1}{2}\frac{\partial R_{js}g^{ls}}{\partial x^i}$.

Theorem 3. (Bolsinov-Matveev) $L = g^{-1}R$ is Nijenhuis operator.

We will discuss about some examples of geodesically (projectivaly) equivalent metrics.

REFERENCES

1. *A. Bolsinov, V. Matveev, A. Konyaev,* Nijenhuis Geometry, Advances in Mathematics, Advances in Mathematics, 394 (2022) 108001.
2. *A. Bolsinov, V. Matveev, A. Konyaev,* Nijenhuis Geometry III: gl-regular Nijenhuis operators, arXiv:2007.09506v1 [math.DG], 18 Jul 2020.
3. *D. Akpan,* Singularities of two-dimensional Nijenhuis operators, European Journal of Mathematics, 8 (2022) 1328–1340.
4. *D. Akpan,* Almost Differentially Nondegenerate Nijenhuis Operators, Russian Journal of Mathematical Physics, 29 (2022) 413-416.
5. *A.V.Bolsinov, V.S.Matveev,* Geometrical interpretation of Benenti systems, Journal of Geometry and Physics 44(2003), no. 4, 489–506.

CONSTRUCTION OF THE ASYMPTOTICS OF A PERIODIC SOLUTION OF THE MACKEY–GLASS EQUATION

V. V. Alexeev[1], M. M. Preobrazhenskaia[2]

[1] *Centre of Integrable Systems, Yaroslavl State University, Yaroslavl, Russia;* vladislav.alexeev.yar@gmail.com

[1] *Centre of Integrable Systems, Yaroslavl State University, Yaroslavl, Russia;* rita.preo@gmail.com

We consider the well-known [1] Mackey–Glass equation which, after renormalizations and substitutions, can be reduced to the differential equation

$$\dot{u} = -\beta u + \frac{\alpha u(t-1)}{1 + (u(t-1))^{\gamma}},$$

with delay 1 and positive parameters $\alpha, \beta, \gamma > 0$. Exponential substitution $u = \mathrm{e}^x$ brings this equation to the following

$$\dot{x} = -\beta + \alpha \frac{\mathrm{e}^{x(t-1)-x}}{1 + \mathrm{e}^{\gamma x}}. \tag{1}$$

Assuming that γ is a large parameter, we obtain the limit equation

$$\dot{x} = -\beta + \alpha \mathrm{e}^{x(t-1)-x} H(\mathrm{e}^{x(t-1)}), \tag{2}$$

where

$$H(u) = \begin{cases} 0, & u > 1, \\ 1/2, & u = 1, \\ 1, & u > 1. \end{cases}$$

Under certain restrictions on the parameters, equation (2) has a periodic solution. In this work we are looking for asymptotic formulas for the periodic solution of equation (1). At the same time, we prove that the resulting solution is close to solution of the limit equation (2), and the remainder is of order $O(\gamma^{-\nu})$, where $\nu \in (1, 1/2)$ is some fixed parameter.

The work of V. V. Alekseev was supported by the Russian Science Foundation (project No. 21-71-30011). The study of M. M. Preobrazhenskaia was funded by a grant Russian Science Foundation No. 22-11-00209, https://rscf.ru/project/22-11-00209/.

REFERENCES

1. *Mackey M. C., Glass L.,* "Oscillation and chaos in physiological control systems," Science, **197**, No. 4300, 287–289 (1977).

FEATURES OF THE DYNAMICS OF A COMPLEX SPATIALLY DISTRIBUTED LOGISTIC EQUATION WITH DELAY

S. V. Aleshin[1], **D. S. Glyzin**[2]

[1]*Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia;* fktiby@yandex.ru

[2]*Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia;* glyzin@gmail.com

We consider a complex logistic equation with diffusion and delay

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + ru(t,x)\Big[1 - |u(t-1,x)|^2\Big], \qquad (1)$$

with periodic boundary conditions

$$u(t,x) \equiv u(t, x + 2\pi), \qquad (2)$$

where complex-valued function $u(t,x)$ is continuously differentiable with respect to $t > 0$ and twice continuously differentiable with respect to $x \in [0, 2\pi]$. Parameter $d \in \mathbb{C}$ and $r \in \mathbb{R}_+$.

The dynamics of the boundary value problem (1), (2) in the case of real variables is well known and has been studied in a large number of works. In article [1] the transition to complex variables was made. Our work represents a continuation and refinement of the numerical analysis presented in [1].

We will assume that $u(t,x) = u_1(t,x) + iu_2(t,x)$, $d = d_1 + id_2$, where $d_1 > 0$. From the boundary value problem (1), (2) we come to the corresponding system for $u_1(t,x)$ and $u_2(t,x)$. As initial conditions for numerical analysis let's take

$$u_j(t,x) = \delta_j + \alpha_j \sin(k_j x - \gamma_j), \quad t \in [-1,0], x \in [0,2\pi], \tag{3}$$

where $\delta_j, \alpha_j, k_j, \gamma_j \in \mathbb{R}$, $j = 1,2$.

For a numerical analysis of solutions to the resulting boundary value problem with the initial condition (3), we replace the second derivative with respect to the spatial variable on the right side of the equation (1) by a finite difference of the second order. In this case, we will divide the segment $[0,2\pi]$ into $K = [2\pi\xi]$ equal parts, where $[.]$ is an integer part of the number, and ξ we will choose one of the values : $100, 200, 250, 400, 500, 1000$ depending on the desired accuracy of calculations.

For calculations, the fifth-order Dorman-Prince method with a variable integration step length was used. The absolute and relative errors of the algorithm were taken equal to 10^{-10}. The initial integration step is taken equal to 10^{-3}.

Numerical analysis showed that the dynamic properties of the boundary value problem (1), (2) differ significantly from the properties of a similar problem with real variables. The most interesting result is the critical dependence of the problem (1), (2) attractors on d_2 (the imaginary part of the "diffusion" parameter). In addition, it was shown that in many cases the resulting stable solutions depend significantly on the frequency of partitioning by the spatial variable and the accuracy of the calculations.

This work was supported by the Russian Science Foundation (project No. 22-11-00209), https://rscf.ru/en/project/22-11-00209/.

REFERENCES

1. *Glyzin, D.S., Kashchenko, S.A.* "Dynamics of the complex spatially distributed Hutchinson equation", Aut. Control Comp. Sci., **48**, 630–632 (2014).

FATEEV–FROLOV–SCHWARZ INSTANTON SUM AND SINGULAR TAU FUNCTION

E. N. Antonov[1], **A. Yu. Orlov**[2]

[1] *Petersburg Nuclear Physics Institute named by B.P.Konstantinov of NRC "Kurchatov Institute", Gatchina, Russia;* antonov@thd.pnpi.spb.ru

[2] *Institute of Oceanology and Higher School of Economics, Moscow, Russia;* orlovs@ocean.ru

In [1] we showed that the Fateev-Frolov-Schwartz (FFS) instanton contribution [2] to the Green's function for the two-dimensional sigma model [3] can be considered as the formal KP tau function, which we obtain using the operator approach (free fermion theory in operator formalism). Namely, it can be chosen as an N-component KP tau function [4], where $N \geq 2$. The inclusion of additional components allows us to consider the variety of instanton correlation functions that are obtained by specification of sets of multicomponent higher times t. The FFS instanton contribution to the generating Green's function of the σ model is an infinite series, each term of which diverges and requires regularization. This brings us to the concept of a regularized tau function [5]. Tau function parameters (known as higher KP times) allow insertions into the fermionic mean to be generated, giving different Green's functions. We use the representation of the instanton partition function Z as a functional integral [6], as was proposed in [7] and use the relation

$$\log Z = V \int \frac{d^2p}{(2\pi)^2} Tr \log(\gamma^\mu p_\mu + m)$$

which can be calculated using dimensional regularization (for instance, see [8]). Here m is the mass of two-dimensional free fermions. We modify this expression to obtain the multicomponent tau function KP, which can be written symbolically as

$$\log Z(t) = V \int \frac{d^2p}{(2\pi)^2} Tr \log(\gamma^\mu p_\mu + m\hat{K}(t))$$

where $\hat{K}(t)$ is an integral operator acting in momentum space and depending on the higher times of the multicomponent KP t (and

$K(0) = 1$) and admitting the same regularization procedure. Finite answers for the Green's functions $\log \frac{Z(t)}{Z(0)}$ are obtained. Examples of correlation functions in the instanton sector are considered.

REFERENCES

1. *A.N. Antonov, A.Yu. Orlov,* "TInstantons in σ-model and tau functions," arXiv preprint arXiv:1611.02248.
2. *V. A. Fateev, I. V. Frolov, A. S. Schwarz,* "Quantum Fluctuations of Instantons in the Nonlinear σ model," Nuclear Physics B, **154**, No. 1, 1–20 (1979).
3. *A. A. Belavin and A. M. Polyakov,* "Metastable states of two-dimensional isotropic ferromagnets," Pis'ma Zh. Eksp. Teor. Fiz., **22**, No. 10, 50–506 (1975).
4. *M. Jimbo, T. Miwa,* "Solutons and Infinite Dimensional Lie Algebra," Publ. RIMS Kyoto Univ, **19**, 943–1000 (1983).
5. *A. N. Antonov, A. Yu. Orlov,* "Sigma model instantons and singular tau function," Journal, in preparation (2023).
6. *A. N. Antonov, A. Yu. Orlov,* "Fateev-Frolov-Schwarz instanton sum and regularized tau function," in preparation (2023).
7. *A. M.Polyakov, "Gauge fields and strings," Landau Institute of Physics and Technology 1995.*
8. *M. E. Peskin and D. V. Schroeder,* "An Introduction to Quantum Field Theory," Journal, Ch 7.5 Perseus Books, Cambridge, Massachusetts.

THE METHOD OF COMPUTING LYAPUNOV EXPONENTS FOR RELAY DIFFERENTIAL EQUATIONS WITH DELAY

D. D. Bain

P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;
danila.bain@yandex.com

Let the sets $P_1, \ldots, P_k$ form a partition of the set $\mathbb{R}^d$, that is, these sets are disjoint and $P_1 \cup P_2 \cup \ldots \cup P_k = \mathbb{R}^d$. Now consider the delay differential equation

$$\dot{u}(t) = f(u,t) + R\big(u(t-T)\big), \tag{1}$$

where $u(t) \in \mathbb{R}^d$, $\quad f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is continuous and Lipschitz with respect to u, and $R(u) = r_i$, if $u \in P_i$, $i = \overline{1, \ldots, k}$. That concludes the definition of the relay DDE.

Choosing appropriate initial conditions, and solving this equation by the method of steps, one can notice that for the values of t such that $u(t-T) \in P_i$ we obtain the solution $u(t)$ as the solution of the differential equation

$$\dot{u}(t) = f(u,t) + r_i. \tag{2}$$

Suppose that we can solve equation (2) exactly. Then, solving equation (1) comes down to finding the values of t at which $u(t-T)$ leaves the set P_i and enters some other set P_j. Finding these switching points exactly is not always possible, but one can use an appropriate numerical root finding algorithm to achieve machine-precision approximations of them. We provide a general numerical algorithm for constructing the solution of equation (1) that uses these ideas. Note that the common iterative algorithms (like Runge–Kutta methods) are not easily applicable since system is discontinuous.

Lyapunov exponents are widely used in studying the dynamics of time-delayed systems, since they are very powerful tools for asymptotic stability analysis and studies about chaotic attractors in nonlinear DDEs. Our goal is to derive an algorithm for computing Lyapunov spectrum for solutions of equation (1) using Benettin method [2]. We start with a particular solution $u(t)$ of equation (1) with initial condition $\varphi(t)$ ($t \in [-T; -1]$) and we consider another solution $\hat{u}(t) = u(t) + \varepsilon w(t)$ of equation (1) with perturbed initial condition $\varphi(t) + \varepsilon\psi(t)$. We substitute $\hat{u}(t)$ into equation (1) and as ε approaches zero we obtain a *linear* equation for $w(t)$ that can be used for Benettin method:

$$\dot{w} = f'_u(u(t),t)w + \left(\sum_{\substack{i,j=1 \\ i<j}}^{k} \delta\Big(D_{ij}\big(u(t-T)\big)\Big)(r_j - r_i) \frac{\Big(w(t-T), N_{ij}\big(u(t-T)\big)\Big)}{\Big|\Big(\dot{u}(t-T), N_{ij}\big(u(t-T)\big)\Big)\Big|} \right),$$

where $D_{ij}(u) = \begin{cases} 0, & \text{if } u \in \partial P_i \cap \partial P_j, \\ 1, & \text{otherwise,} \end{cases}$ and $N_{ij}(u)$ is a unit vector normal to the intersection of boundaries $\partial P_i \cap \partial P_j$ directed towards P_j.

The rest of the work is devoted to proving necessary conditions for this method to be applicable and to work through examples of asymptotically stable cycles as well as examples of chaotic oscillations to show how the method is applied in practice.

REFERENCES

1. *Lian, Zeng* "Lyapunov Exponents and Invariant Manifold for Random Dynamical Systems in a Banach Space" (2008). Theses and Dissertations. 1517.
2. *Bennetin, G., Galgani, L., Giorgilli, A., and Strelcyn, J.M.* Lyapunov characteristic exponents for smooth dynamical systems and for hamiltonian systems; a method for computing all of them. part 2: Numerical applications. Meccanica, 15, 9–20, 21–30.

PIECEWISE-SMOOTH MODELING OF LORENZ-TYPE SYSTEMS

D. A. Bakalina[1], **N. V. Barabash**[1,2]

[1] *Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, Russia;* darya.bakalina@gmail.com

[2] *Volga State University of Water Transport, Nizhny Novgorod, Russia;* barabash@itmm.unn.ru

In this talk we present a numerical comparison of the scenarios for the birth of attractors in a piecewise-smooth system of the Lorenz type [1-3] with Lorenz system [4] and Lyubimov–Zaks system [5]. The piecewise-smooth system A is formed by three linear subsystems A_s, A_l and A_r.

$$A_s: \begin{array}{l} \dot{x} = x, \\ \dot{y} = -\alpha y, \\ \dot{z} = -\nu z, \end{array} \quad A_{l,r}: \begin{array}{l} \dot{x} = -\lambda(x \pm 1) + \omega(z - b), \\ \dot{y} = -\delta(y \pm 1), \\ \dot{z} = -\omega(x \pm 1) - \lambda(z - b) \end{array}$$

where α, δ, ν, ω, λ and b are positive parameters. Systems $A_{s,l,r}$ are defined in the following partitions of the system's phase space $G_s = \{|x| < 1, y \in \mathbb{R}^1, z < b, \}$, $G_l = \{x \leq -1, z \leq b; x \leq -1, z > b, y \leq 0, x < 1, z > b, y < 0\}$, $G_r = \mathbb{R}^3 \setminus \{G_s \cup G_l\}$, respectively.

We show that, depending on ν, which is a saddle index of the saddle at the origin, two different bifurcation scenarios are observed in system A. For $\nu < 1$, a scenario similar to the transition to chaos in the Lorenz system is realized. It is a classical scenario of the birth of a singular-hyperbolic Lorenz-type attractor, which passes through a heteroclinic bifurcation of the saddle's separatrices and two saddle limit cycles. For $\nu > 1$, the transition to chaos occurs through a cascade of alternating bifurcations of the homoclinic orbits of the saddle, and pitchfork bifurcations of stable limit cycles, leading to a period doubling and orbits doubling. This scenario is similar to the one in Lyubimov–Zaks system.

Qualitatively and numerically we demonstrate a similarity of system A to the original Lorenz system and Lorenz-type Lyubimov–Zaks system in terms of their main properties and scenarios for the birth of attractors. However, unlike its smooth counterparts, the piecewise-smooth system A allows carry out a complete analytical study including the proof of the strange attractor's existence and global bifurcations.

The work was supported by the Ministry of Science and Higher Education (Project No. 0729-2020-0036).

REFERENCES

1. *Belykh V. N., Barabash N. V., Belykh I. V.* A Lorenz-type attractor in a piecewise-smooth system: Rigorous results // Chaos: An Interdisciplinary Journal of Nonlinear Science, **29**, No. 10, 103108 (2019).
2. *Belykh V. N., Barabash N. V., Belykh I. V.* Bifurcations of chaotic attractors in a piecewise smooth Lorenz-type system //Automation and Remote Control, **81**, No. 8, 1385–1393 (2020).
3. *Belykh V. N., Barabash N. V., Belykh I. V.* Sliding homoclinic bifurcations in a Lorenz-type system: Analytic proofs //Chaos: An Interdisciplinary Journal of Nonlinear Science, **31**, No. 4, 043117 (2021).
4. *Lorenz E. N.* Deterministic nonperiodic flow //Journal of atmospheric sciences, **20**, No. 2, 130–141 (1963).
5. *Lyubimov D. V., Zaks M. A.* Two mechanisms of the transition to chaos in finite-dimensional models of convection //Physica D: Nonlinear Phenomena, **9**, No. 1–2, 52–64 (1983).

NONLINEARITY IN INVERSE PROBLEMS OF ORBITAL DYNAMICS ON THE EXAMPLE OF POTENTIALLY HAZARDOUS ASTEROIDS AND JUPITER'S OUTER SATELLITES

M. A. Banshchikova[1], **O. M. Syusina**[2]

[1] *Tomsk State University, Russia;* mba-tomsk@mail.ru
[2] *Tomsk State University, Russia;* olga_syusina@rambler.ru

The paper investigates the nonlinearity in inverse problems of the orbital dynamics of potentially hazardous asteroids and outer Jupiter's satellites discovered from 1999 to 2018. As is known, in order to find object's orbital parameters, it is necessary to solve the inverse problem.

For this, a system of nonlinear equations (1) is compiled. To solve it, the least squares method is used.

$$\mathbf{p}^O = \mathbf{p}^C(\mathbf{q}); \tag{1}$$

where $\mathbf{p}^O$ and $\mathbf{p}^C$ are N-dimensional vectors of observed and calculated quantities, respectively.

When finding the parameter estimate the least-squares principle reduces to minimizing the objective function F of the form (2).

$$\hat{\mathbf{q}} : \Phi(\hat{\mathbf{q}}) = \|\mathbf{p}^O - \mathbf{p}^C(\mathbf{q})\|^2 \to 0. \tag{2}$$

Due to the presence of various errors, including observation errors, we cannot find the exact value of the parameters. Therefore, it is necessary to build a cloud of uncertainty that will contain this exact value. Various methods can be used to build a cloud of uncertainty (Milani, 1999; Muinonen et al., Avdyushev, 2022). Construction methods can be divided into two large classes - linear and non-linear. Linear methods are quite simple, easy to implement and do not require large computational and time costs. A cloud of uncertainty constructed using such methods will always have an ellipsoidal shape. In the case of a strong nonlinearity in the estimation problem, the real cloud of uncertainty will deviate from the ellipsoidal shape and we will get an unreliable result. To understand whether linear methods can be applied, one must first evaluate the nonlinearity of the estimation problem. This can be done using non-linearity indicators (Bates, 1980; Beale, 1960; Syusina, 2012).The nonlinearity can be intrinsic and parametric. Intrinsic non-linearity is characteristic of a particular problem and cannot be eliminated. Parametric nonlinearity is a consequence of the influence of external factors. In a number of cases, such nonlinearity can be reduced by an appropriate choice of the parametric space. The work (Avdyushev, 2021) proposes original indicators that allow one to determine the intrinsic and parametric nonlinearities of the inverse problem.

The objects of study in this paper are potentially hazardous asteroids and Jupiter's outer satellites. Among potentially hazardous asteroids, only those objects that were observed in one appearance were selected. Such asteroids are most interesting for research due to the short time intervals of observations. Among the outer satellites of Jupiter, only those discovered from 1999 to 2018 were selected. The selected outer satellites of Jupiter are characterized by irregular orbits, significant perturbations from the Sun, and a meager and unevenly distributed set of observations. These circumstances lead to a significant nonlinearity at the solution of inverse problems of orbital dynamics. On the example of these objects, the nonlinearity of the estimation problem in the Cartesian and Keplerian spaces was estimated. In addition, a study was made of the influence of the initial epoch choice on the nonlinearity indicator's value. It is shown that for all objects the nonlinearity degree of the inverse problem depends on many factors, such as: the used parametric space; the size of the interval covered by the observations; the nature of the distribution of observations on the measured interval; the initial epoch chosen for solving the inverse problem. But in the case of studying the outer satellites of Jupiter or asteroids, there are a number of features.

When solving problems of asteroid dynamics for objects with a small observation arc, it is advisable to choose the space of Cartesian variables, in which the nonlinearity of the problem is less than in the Keplerian space. For objects observed over a long time interval and in several appearance, the nonlinearity, both in Cartesian and Keplerian space, has approximately one order per epoch, equal to the arithmetic mean of all observation moments. However, in the space of Cartesian variables, the choice of the initial epoch outside the observation arc leads to a rapid increase in the nonlinearity with its removal from the epoch, which is equal to the arithmetic mean of all observation moments. At the same time, in the space of Keplerian variables, the nonlinearity grows rather weakly in this case.

When solving inverse problems of satellite dynamics, it is shown that if one group of satellite observations is used for the solution, or a large number of uniformly distributed observations, then less nonlinearity is achieved in the Kepler parametric space. Cartesian parametric space becomes more preferable if two/three groups of observations are used in solving the inverse problem, the distance between which exceeds half of the satellite's orbital period. The nonlinearity, both in Keplerian and Cartesian space, can be further reduced by searching for a suitable initial epoch inside the dimensional interval.

Thus, for each object, the nonlinearity was estimated in two parametric spaces — Cartesian and Keplerian. The optimal initial epochs providing the least nonlinearity in the considered parametric spaces are determined. It is shown which of the parametric spaces, together with the optimal initial epoch, is more preferable to achieve the least nonlinearity in each specific case.

The research was carried out within the state assignment of Ministry of Science and Higher Education of the Russian Federation (theme No. FSWM-2020-0049). The work was carried out using the equipment of the Center for collective use Terskol observatory.

REFERENCES

1. *Milani A.* "The asteroid identification problem I: Recovery of lost asteroids.", Icarus No. 137, 269–292 (1999).
2. *Muinonen K., Virtanen J., Granvik M., Laakso T.* "Asteroid orbits using phase-space volumes of variation.," Mon. Not. R. Astron. Soc., No. 368, 809–818 (2006).
3. *Avdyushev V. A.* "Piecewise transition methods for stochastic simulation of uncertainty in orbital parameters.," Cel. Mech. Dyn. Astr., No. 134(6), (2022).
4. *Bates D.M., Watts D.G.* "Relative Curvature Measures of Nonlinearity" J. R. Statist. Astr., V. 42, No.1, 1–25 (1980).
5. *Beale E.M.L.* "Confidence Regions in Nonlinear Estimation" J. R. Statist. Astr., V. 22, 41–88. (1960).
6. *Syusina, O.M., Chernitsov, A.M., Tamarov, V.A.* "Construction of confidence regions in problem on probabilistic study into motion of minor bodies of the solar system" Sol. Sys. Res., V. 46, No.3, 195–207. (2012).
7. *Avdyushev V.A., Syusina O.M., Tamarov V.A.* "Nonlinearity in inverse problems of asteroid dynamics.," Sol. Sys. Res., No. 55, 71–82 (2021).

PERIODIC SMALE HORSESHOES UNCOVER TRUE COMPLEXITY OF DOUBLE SCROLL ATTRACTOR

N. V. Barabash[1,2], **V. N. Belykh**[1,2], **I. V. Belykh**[3]

[1]*Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, Russia;* barabash@itmm.unn.ru

[2]*Volga State University of Water Transport, Nizhny Novgorod, Russia;*

[3]*Georgia State University, Atlanta, USA;* ibelykh@gsu.edu

In this report, we consider new objects of chaos theory called periodic Smale horseshoes [1], representing a chain of sequentially linked images and preimages of different topological horseshoes of some nonlinear map. We show that these horseshoes defined previously unknown significant complexity of an attractor with odd-symmetric returnability of the flow to a sadlle-focus with 2-D stable and 1-D unstable manifolds (a double scroll-attractor).

We proved the existence of these periodic horseshoes for a double-scroll attractor of a piecewise-smooth system A of the form:

$$A_0: \begin{array}{l} \dot{x} = x, \\ \dot{y} = -\nu y + \omega z, \\ \dot{z} = -\omega y - \nu z, \end{array} \quad A_{l,r}: \begin{array}{l} \dot{x} = -\alpha(x \pm h) - \Omega(z+1), \\ \dot{y} = -\beta y, \\ \dot{z} = \Omega(x \pm h) - \alpha(z+1) \end{array}$$

where α, h, ν, ω, Ω and β are positive parameters, and regions $G_{0,l,r}$ are defined as follows: $G_0 = \{|x| < h, (y^2 + z^2 \leq r^2) \cap (|z| < 1)\}$, $G_l = \{(z \leq -\text{sign } x, y \in \mathbb{R}^1) \setminus G_0\}$ and $G_r = \{(z \geq -\text{sign } x, y \in \mathbb{R}^1) \setminus G_0\}$.

Using our method of piecewise-linear reconstruction of nonlinear systems [2-4], we obtained the Poincaré return map for this system, for which we derived conditions for the existence of periodic Smale horseshoes of arbitrary period. We discovered the infinite chain of Smale horseshoes and established their crucial role in the formation of the attractor. We show that the invariant set of the attractor and its structure are determined precisely by periodic Smale horseshoes and infinite chains, rather than by the classical Shilnikov theorem, which is related only to a small part of this set in a small neighborhood of the saddle-focus.

The authors were supported by the Ministry of Science and Higher Education (project No. 0729-2020-0036).

REFERENCES

1. *Belykh V. N., Barabash N. V., Belykh I. V.* The hidden complexity of a double-scroll attractor: Analytic proofs from a piecewise-smooth system // Chaos, **33**, No. 4, 043119 (2023).
2. *Belykh V. N., Barabash N. V., Belykh I. V.* A Lorenz-type attractor in a piecewise-smooth system: Rigorous results // Chaos, **29**, No. 10, 103108 (2019).
3. *Belykh V. N., Barabash N. V., Belykh I. V.* Bifurcations of chaotic attractors in a piecewise smooth Lorenz-type system //Automation and Remote Control, **81**, No. 8, 1385–1393 (2020).
4. *Belykh V. N., Barabash N. V., Belykh I. V.* Sliding homoclinic bifurcations in a Lorenz-type system: Analytic proofs //Chaos, **31**, No. 4, 043117 (2021).

ON ORBITAL STABILITY OF PENDULUM-LIKE PERIODIC MOTIONS OF A HEAVY RIGID BODY IN A TRANSCENDENTAL CASE

B. S. Bardin

Moscow Aviation Institute (National Research University), Moscow, Russia; bsbardin@yandex.ru

The problem of orbital stability of planar periodic motions of a heavy rigid body with a fixed point is studied. The mass geometry of the body corresponds to the Hess case [1]. It was considered two types of the periodic motions: pendulum-like oscillations and pendulum-like rotations.

The system of equations describing the perturbed motion in a neighborhood of pendulum-like oscillations and rotations was written in a Hamiltonian form. By analyzing the linearized system it has been proven that the pendulum-like rotations of the body are orbitally unstable for any values of parameters. The problem of orbital stability of the pendulum-like oscillations is much more difficult. It was shown that depending on values of parameters the pendulum-like oscillations can be both stable or unstable in the linear approximation. However, rigorous conclusions on orbital stability for the pendulum-like oscillations cannot be drawn by means of the linear analysis. Moreover, it was shown that for the pendulum-like oscillations the so-called transcendental case [2] takes place, that is the orbital stability problem cannot be also solved by means of the study of a non-linear approximation of any finite order. By using the method developed in [3] the orbital instability of the pendulum-like oscillations has been proven.

This research was supported by the grant of the Russian Science Foundation (project No. 22-21-00729) and was carried out at the Moscow Aviation Institute (National Research University).

REFERENCES

1. *Hess, W.*, "Ueber die Euler'schen Bewegungsgleichungen und über eine neue partikuläre Lösung des Problems der Bewegung eines starren Körpers um einen festen Punkt," Math. Ann., **37**, No. 2, 153–181 (1890).
2. *Lyapunov A. M.*, "Principles," in: Collected Works, Akad. Nauk SSSR, Moscow, 1956, Vol. 2, pp. 264–271 (in Russian).
3. *Bardin B. S.*, "On the Stability of a Periodic Hamiltonian System with One Degree of Freedom in a Transcendental Case," Doklady Mathematics, **97**, No. 2, 161–163 (2018).

AN ANALOGUE OF THE GENERALIZED JACOBI AND CHASLES THEOREM FOR SPACES OF CONSTANT GAUSSIAN CURVATURE

G. V. Belozerov

Lomonosov MSU, Moscow, Russia; gleb0511beloz@yandex.ru

According to the classical Jacobi and Chasles theorem, the geodesic flow on a quadric in n-dimensional Euclidean space is a quadratically integrable Hamiltonian system (see [1]–[2]). As it turned out, a similar result holds for a geodesic flow at the intersection of several confocal quadrics.

Theorem 1. [Belozerov] Let $Q_1, \ldots, Q_k$ be non-degenerate confocal quadrics in n-dimensional Euclidean space and $Q = \bigcap\limits_{i=1}^{k} Q_i$. Then

1. the geodesic flow on Q is a quadratically integrable Hamiltonian system.
2. the tangent lines drawn at all points of a geodesic curve on Q are tangent, as well as to the given quadrics $Q_1, \ldots, Q_k$, to $n-k-1$ other confocal quadrics, which are the same for all points of the geodesic curve.

Remark. Note that the case $k = n - 2$ was proved earlier by Kibkalo V. A.

Based on this theorem, academician A. T. Fomenko hypothesized that Jacobi and Chasles theorem does not hold only for Euclidean spaces, but also for spaces of constant Gaussian curvature (namely, for n-dimensional sphere $\mathbb{S}^n$ and n-dimensional Lobachevsky space $\mathbb{L}^n$). As it turns out, this hypothesis is true.

The report will be devoted to an analogue of theorem 1 for spaces of constant Gaussian curvature. We will define confocal quadrics in these spaces, find out their properties and discuss the proof of the following theorem.

Theorem 2. [Belozerov, Fomenko] Let $Q_1, \ldots, Q_k$ be non-degenerate confocal quadrics in $\mathbb{S}^n$(or in $\mathbb{L}^n$) and $Q = \bigcap\limits_{i=1}^{k} Q_i$. Then

1. the geodesic flow on Q is a quadratically integrable Hamiltonian system.

2. the tangent lines in terms of $\mathbb{S}^n$(or in $\mathbb{L}^n$) drawn at all points of a geodesic curve on Q are tangent, as well as to the given quadrics $Q_1, \ldots, Q_k$, to $n-k-1$ other confocal quadrics, which are the same for all points of the geodesic curve.

REFERENCES

1. *Jacobi C. G. J.*, Vorlesungen über Dynamik, 2. rev. Ausg., GesammelteWerke, Supplementband, G. Reimer, Berlin, 1884, viii+300 pp..
2. *Chasles M.*, "Sur les lignes géodésiques et les lignes de courbure des surfaces du second degré," J. Math. Pures Appl., 11 (1846), 5–20.
3. *Belozerov G. V.*, "Integrability of a geodesic flow on the intersection of several confocal quadrics," Dokl. Math., 107:1 (2023), 1–3.

ON ATTRACTORS OF CERTAIN DYNAMICAL SYSTEMS

V. N. Belykh

Volga State University of Water Transport, Nizhny Novgorod, Russia; Lobachevsky State University of Nizhny Novgorod, Nizhny Novgorod, Russia; belykh@unn.ru

In this talk we consider various attractors of certain dynamical systems that satisfy the condition for the existence of a maximal attractor

$$f^+D \subset D, \quad A = \bigcap_{t>0} f^t D,$$

where D is a compact, f^t is either a flow for $t \in \mathbb{R}^1$, or a map for $t \in \mathbb{Z}$. Examples of attractors of the Hénon map, Lorenz-type systems, and systems with a saddle-focus with two homoclinic orbits (double scroll) are given.

The work was supported by the Ministry of Science and Higher Education (Project No. 0729-2020-0036).

PHASE DYNAMIC ANALYSIS OF EEG AND CARDIAC SIGNALS IN COMATOSE PATIENTS

B. P. Bezruchko[1,3], **E. V. Navrotskaya**[1,3], **R. S. Khambekov**[2], **E. E. Zeulina**[2]

[1]*Saratov State University, Saratov, Russia;* bezruchkobp@gmail.com
[2]*Saratov State Medical University, Saratov, Russia;* zeulina@list.ru
[3]*Saratov Branch of the Institute of Radio Engineering and Electronics of Russian Academy of Sciences;* sidakev@gmail.com

The diagnostics of interaction between various physiological processes in humans is carried out using time series of different biological signals (electrocardiogram, electroencephalogram, myogram, etc). Many of such signals can be recorded noninvasively that simplifies the possibility of their analysis in clinical practice. In recent decades new knowledge has been obtained about the functioning of different elements of human cardiovascular system (CVS). In particular, it was shown that measurements of phase synchronization between the low-frequency rhythms (with a frequency of about 0.1 Hz) in heart rate variability (HRV) and photoplethysmogram (PPG) signal can help in choosing the more efficient therapeutic approach for treatment of several CVS diseases [1,2]. However, it is important to apply other quantitative measures for studying the features and structure of couplings between different physiological processes especially in the cases of intensive therapy and reanimation. In this work we investigate the characteristics of coupling between slow rhythms of EEG, autonomic control of heart rate and peripheral blood filling (from PPG data) in patients with depressed level of consciousness. The nonlinear methods developed for the analysis of couplings between oscillators were applied to the study of electroencephalogram(EEG), electrocardiogram (ECG) and photoplethysmogram (PPG) signals from patients in coma with cerebrovascular accident. The registered signals were eight channels of EEG, the single-channel ECG at the II standard lead and three channels of PPG (from earlobe, right ring finger, and right second toe). The signals were filtered in different bands, including the 0.05–0.15 Hz which includes the rhythms of autonomic control of heart rate and vascular tone. For coupling analysis, we used the method based on the phase dynamics modelling [3,4]. We used the phases as characteristics of oscillations in cardio-vascular system because the phases are the most sensitive to changes in the systems parameters. The analysis of phase dynamics of the studied physiological oscillatory systems revealed that cerebrovascular accident during coma suppresses the coupling between the rhythms of autonomic control of heart rate (extracted from ECG), vascular tone (extracted from PPG) and slow rhythms of EEG.

This study was funded by the Russian Science Foundation, Grant No. 23-12-00241.

REFERENCES

1. *Prokhorov M.D., Ponomarenko V.I., Gridnev V.I., Bodrov M.B., Bespyatov A.B.* "Synchronization between main rhythmic processes in the human cardiovascular system", Phys Rev E Stat Nonlin Soft Matter Phys, **68**, 041913 (2003).
2. *Karavaev A.S., Prokhorov M.D., Ponomarenko V.I., Kiselev A.R., Gridnev V.I., Ruban E.I., et al.* "Synchronization of low-frequency oscillations in the human cardiovascular system", Chaos, **19(6)**, 033112 (2009).
3. *Smirnov D.A., Bezruchko B.P.* "Estimation of interaction strength and direction from short and noisy time series", Phys. Rev. E Stat. Nonlin. Soft. Matter. Phys., **68**, 046209 (2003).
4. *M.G. Rosenblum, A.S. Pikovsky* "Detecting direction of coupling in interacting oscillators", Phys. Rev. E., **64**, 041913 (2001).

DIFFERENTIAL-ALGEBRAIC THEORY OF MUMFORD DYNAMICAL SYSTEM AND APPLICATIONS

V. M. Buchstaber

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia; buchstab@mi-ras.ru

Based on theta-functional solutions of the g-Novikov equation and the g-stationary Korteweg-de Vries hierarchy Dubrovin and Novikov showed that the universal space of Jacobians of hyperelliptic curves of genus g is birationally equivalent to the space $\mathbb{C}^{3g+1}$, $g \geqslant 1$.

Based on the theory of non-singular hyperelliptic curves and the theta-functional theory of their Jacobians, Mumford introduced the dynamical system

$$\begin{aligned}\mathcal{D}_\eta u_\xi =& \frac{2}{\xi-\eta}(v_\xi u_\eta - u_\xi v_\eta),\\ \mathcal{D}_\eta v_\xi =& \frac{1}{\xi-\eta}(u_\xi w_\eta - w_\xi u_\eta) + u_\xi u_\eta,\\ \mathcal{D}_\eta w_\xi =& \frac{2}{\xi-\eta}(w_\xi v_\eta - v_\xi w_\eta) - 2v_\xi u_\eta\end{aligned}$$

in the space $\mathbb{C}^{3g+1}$ with coordinates $\mathbf{u} = (u_1, \ldots, u_g)$, $\mathbf{v} = (v_1, \ldots, v_g)$, $\mathbf{w} = (w_1, \ldots, w_{g+1})$, where ξ and η are independent parameters, $\mathcal{D}_\eta = \sum\limits_{i=1}^{g} \eta^{g-i}\partial_i$, $\partial_i = \frac{\partial}{\partial t_i}$, $\mathcal{D}_\eta \xi = \mathcal{D}_\eta \eta = 0$ and

$$u_\xi = \xi^g + \sum_{i=1}^{g} u_i \xi^{g-i}, \quad v_\xi = \sum_{i=1}^{g} v_i \xi^{g-i}, \quad w_\xi = \xi^{g+1} + \sum_{i=1}^{g+1} w_i \xi^{g+1-i}.$$

This system has a Lax representation and, as a consequence, obtains $2g+1$ of the integral $h_1, \ldots, h_{2g+1}$ with generating function

$$H_\xi = u_\xi w_\xi + v_\xi^2 = \xi^{2g+1} + \sum_{n=1}^{2g+1} h_n \xi^{2g+1-n}.$$

Within the framework of the algebraic-geometric theory of integrable systems, Vanhaecke and a number of other authors developed Mumford's theory of dynamical systems and its generalizations.

In this talk we will present the differential algebraic theory of this system and its applications in the theory of the Korteweg-de Vries hierarchy.

Theorem 1 *For each infinitely differentiable function $u_1 = u_1(t_1)$ and any vector $\mathbf{h} = (h_1, h_2, \ldots)$ is uniquely determined an infinite sequence of differential polynomials, where $\mathbf{h}^k = (h_1, \ldots, h_k)$, $k = 1, \ldots,$ $f' = \partial_1 f$,*

$$P_k = P_k(u_1, u_1', \ldots, u_1^{(2k-2)}; \mathbf{h}^k), \quad Q_k = Q_k(u_1, u_1', \ldots, u_1^{(2k-2)}; \mathbf{h}^k),$$

such that: $P_1 = u_1$, $Q_1 = h_1 - u_1$; $P_2 = \frac{1}{4}(u_1'' + 6u_1^2 - 4h_1 u_1 + 2h_2)$, $Q_2 = \frac{1}{4}(-u_1'' - 2u_1^2 + 2h_2)$. Polynomials P_k and Q_k, $k \geqslant 3$, are *given recurrently by the formulas:*

$$\begin{aligned} P_k =& \frac{1}{4}P''_{k-1} - \frac{1}{2}\sum_{i+j=k} P_iQ_j - \\ & - \frac{1}{8}\sum_{i+j=k-1} P''_iP''_j - \frac{1}{2}(h_1 - 2u_1)P_{k-1} + \frac{1}{2}h_k; \end{aligned} \tag{1}$$

$$\begin{aligned} Q_k =& -\frac{1}{4}P''_{k-1} - \frac{1}{2}\sum_{i+j=k} P_iQ_j - \\ & - \frac{1}{8}\sum_{i+j=k-1} P''_iP''_j + \frac{1}{2}(h_1 - 2u_1)P_{k-1} + \frac{1}{2}h_k. \end{aligned} \tag{2}$$

We will call the recursion, which according to formulas (1) and (2) gives a sequence of pairs of differential polynomials (P_k, Q_k), $k = 1, 2, \ldots,$ from the function $P_1 = u$, a (P, Q)-recursion. It follows directly from the construction of (P, Q)-recursion and the results of [1]:

Theorem 2 *(P, Q)-recursion gives an infinite system of simultaneous differential equations $\partial_k P_1 = \partial_1 P_k$, where ∂_k, $k = 1, 2, \ldots$, is a sequence of commuting derivations of the ring $\mathbb{C}[u_1, u_1', \ldots; \mathbf{h}]$, $\partial_k h_i = 0$.*

Corollary 1 *The general solution of the (P, Q)-recursion gives the general solution of the parametric Korteweg-de Vries hierarchy.*

An explicit formula relating these solutions will be given.

Theorem 3 *Let $(\mathbf{u} = \mathbf{u}(\mathbf{t}), \mathbf{v} = \mathbf{v}(\mathbf{t}), \mathbf{w} = \mathbf{w}(\mathbf{t}))$, where $\mathbf{t} = (t_1, \ldots, t_g) \in \mathbb{C}^g$, a solution of the Mumford system, regular in a neighborhood of the point $\mathbf{t}^*$. Let $P_k, k = 1, \ldots, g$, and $Q_k, k = 1, \ldots, g+1$, be differential polynomials 1. Then:*
1. *The condition $P_{g+1} = 0$ gives the equation*

$$P''_g = 2(h_1 - 2u_1)P_g + 2\sum_{i+j=g+1} P_iQ_j + \frac{1}{2}\sum_{i+j=g} P''_iP''_j - 2h_{g+1}, \tag{3}$$

which is a nonlinear ordinary differential equation of order $2g$ on $u_1 = u_1(\mathbf{t})$ as a function of t_1. A solution to the equation (3) in the vicinity of the point $t_1^* \in \mathbb{C}$ exists and is uniquely specified by the initial data $t_1^* \in \mathbb{C}$ exists and is uniquely specified by the initial data

$\mathbf{z} = (z_0, \ldots, z_{2g-1}) \in \mathbb{C}^{2g}$ *and the values of the vector* $\mathbf{h}^{g+1}$ *of the parameters of this equation, where by definition* $z_0 = u_1(\mathbf{t}^*), \ldots, z_{2g-1} = u_1^{(2g-1)}(\mathbf{t}^*)$.

2. *Formulas*

$$u_k(\mathbf{t}) = P_k(\mathbf{t}),\ v_k(\mathbf{t}) = P_k'(\mathbf{t}),\ k = 1, \ldots, g;$$

$$w_k(\mathbf{t}) = Q_k(\mathbf{t}),\ k = 1, \ldots, g+1,$$

where $P_1(\mathbf{t}) = u_1(\mathbf{t})$, $Q_1(\mathbf{t}) = h_1 - u_1(\mathbf{t})$, *define a one-to-one mapping neighbourhood of a point* $(\mathbf{z}^*, \mathbf{h}^{g+1}) \in \mathbb{C}^{2g} \times \mathbb{C}^{g+1}$ *in the vicinity of a point* $(\mathbf{u}(\mathbf{t}^*), \mathbf{v}(\mathbf{t}^*), \mathbf{w}(\mathbf{t}^*)) \in \mathbb{C}^{3g+1}$. *Here* $\mathbf{z}^* \in \mathbb{C}^{2g}$ – *C, point that specifies the initial conditions for solving the equation* (3) *with parameters* $\mathbf{h}^{g+1}$, *which is regular in a neighbourhood of the point* $\mathbf{t}^* \in \mathbb{C}^g$.

Let $\sigma(\mathbf{t})$ be a hyperelliptic sigma function and $\wp_\omega(\mathbf{t}) = -\frac{\partial^n}{\partial t_{i_1} \cdots \partial t_{i_n}} \ln \sigma(\mathbf{t})$, see [2]. Set $\wp_{2k} = \wp_{(1,k)}(\mathbf{t})$ and $x_\xi = \xi^g - \sum_{i=1}^{g} \wp_{2i} \xi^{g-i}$, $y_\xi = \sum_{i=1}^{g} \wp_{2i}' \xi^{g-i}$, $z_\xi = \sum_{i=1}^{g} \wp_{2i}'' \xi^{g-i}$.

Theorem 4 *Functions* $u_\xi = 2x_\xi$, $v_\xi = y_\xi$, $w_\xi = z_\xi + 2x_\xi(\xi + 2\wp_2)$ *give the solution Mumford dynamic system.*

The proof uses an explicit description of all algebraic relations between Abelian functions $\wp_\omega$.

REFERENCES

1. *Baron P. G.*, "Mumford dynamic system and Gelfand–Dikii recursion," Funct. Anal. Appl., **57**, No. 4 (2023).
2. *Buchstaber V. M.*, *Enolskii V. Z.*, *Leikin D. V.*, "leinian functions, hyperelliptic Jacobians and applications," Reviews in Mathematics and Math. Physics, v. 10, part 2, Gordon and Breach, London, 1997, 3–120.
3. *Buchstaber V. M.*, "Mumford dynamical system and hyperelliptic Klein functions," Funct. Anal. Appl., **57**, No. 4 (2023).

NUMERICAL INTEGRATION OF ONE-DIMENSIONED REACTION-DIFFUSION-ADVECTION PROBLEM USING ADAPTIVE MESH

P. E. Bulatov

Lomonosov MSU, Moscow, Russia; bulatov.pe14@physics.msu.ru

Let us consider the one-dimensioned parabolic Neumann problem [1,2]:

$$\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\varepsilon A(x,t)\,\frac{\partial u}{\partial x}-\varepsilon\frac{\partial u}{\partial t}=f(u,x)\,;\quad \left.\frac{\partial u}{\partial x}\right|_{x=0}=h_0;\quad \left.\frac{\partial u}{\partial x}\right|_{x=1}=h_1;$$
$$x\in X:=(0,1)\,,\ t\in(0,+\infty)\,;\quad u(x,0)=u_{init}(x)\,.$$

In case of non-linear right hand (e.g. cubic) the solution consists ***contrast structures*** – thin domains of high gradient. Numerical integration is complicated by the fact that the spatial step must be much less than characteristic size of the contrast structure.

Let's apply the approach [3] and the method of lines. We reduce the original problem to a differential-algebraic system which will be solved via the complex Rosenbrock scheme. The algebraic part is excluded when the solution in internal nodes is found and the Jacoby matrix is expressed analytically. It is tridiagonal and then allows us to apply the non-monotonic tridiagonal matrix algorithm [4].

$$\frac{du_n}{dt}=4B(x,t)\frac{u_{n+1}-2u_n+u_{n-1}}{\left(x_{n+1}-x_{n-1}\right)^2}-\frac{u_{n+1}-u_{n-1}}{x_{n+1}-x_{n-1}}\left\{B(x,t)\frac{x_{n+1}-2x_n+x_{n-1}}{\left(x_{n+1}-x_n\right)\left(x_n-x_{n-1}\right)}-\right.$$
$$\left.-\frac{\hat{x}_n-\check{x}_n}{\hat{t}-\check{t}}\right\}-f(u,x,t)\,,\quad n=\overline{1,N-1};\qquad u|_{t=t_0}=u_{init}(x_n)\,,\quad n=\overline{0,N};$$
$$u_0=\frac{\left(x_2-x_0\right)^2u_1-\left(x_1-x_0\right)^2u_2}{\left(x_2+x_1-2x_0\right)\left(x_2-x_1\right)}-\frac{\left(x_1-x_0\right)\left(x_2-x_0\right)}{x_2+x_1-2x_0}h_l;$$
$$u_N=\frac{\left(x_N-x_{N-2}\right)^2u_{N-1}-\left(x_N-x_{N-1}\right)^2u_{N-2}}{\left(2x_N-x_{N-1}-x_{N-2}\right)\left(x_{N-1}-x_{N-2}\right)}-\frac{\left(x_N-x_{N-1}\right)\left(x_N-x_{N-2}\right)}{2x_N-x_{N-1}-x_{N-2}}h_r.$$

The new-layer mesh is built iteratively via [5]:

$$\hat{x}_n^{s+1}=(1-\sigma)\,\hat{x}_n^s+\sigma\left(\sum_{p=0}^{N-1}\omega^{-1}_{p+\frac12}(\hat{x}^s)\right)^{-1}\sum_{p=0}^{n-1}\omega^{-1}_{p+\frac12}(\hat{x}^s)\,,\quad \omega_{p+\frac12}(\hat{x}^s)=\left|\hat{u}^s_{p+1}-\hat{u}^s_p\right|+\delta.$$

With $\varepsilon=10^{-4}$ and the number of spatial steps $N=2048$ we successfully reduced the step in contrast structures by 160 times. Uniform grids require 327680 steps per layer for the same resolution.

The authors were supported by the Russian Science Foundation (project no. 23-11-00069).

REFERENCES

1. *Butuzov V. F., Nefedov N. N., Schneider K. R.,* "O formirovanii i rasprostranenii rezkikh perekhodnykh sloev v parabolicheskikh zadachakh," Vestnik Moskovskogo universiteta. Seriya 3. Fizika. Astronomiya, No. 1, 9–13 (2005) (in Russian).
2. *Bozhevolnov Yu. V., Nefedov N. N.,* "Dvizhenie fronta v parabolicheskoj zadache reaktsiya-diffuziya," Zhurnal vychislitelnoj matematiki i matematicheskoj fiziki, **50**, No. 2, 276–285 (2010) (in Russian).
3. *Degtyarev L. M., Ivanova T. S.,* "Metod adaptivnykh setok v odnomernykh nestatsionarnykh zadachakh konvektsii-diffuzii," Differentsialnyie uravneniya, **29**, No. 7, 1179–1192 (1993) (in Russian).
4. *Samarskii A. A., Nikolaev E. S.,* Metody resheniya setochnykh uravnenij, Nauka, Moscow (1978) (in Russian).
5. *Pokhilko V. I., Tishkin V. F.,* "Odnorodny algoritm rascheta razryvnykh reschenij na adaptivnykh setkakh," Matematicheskoe modelirovanie, **6**, No. 11, 25–40 (1994) (in Russian).

CLASSIFICATION OF SOLUTIONS TO LOCAL TETRAHEDRON EQUATION

M. A. Chirkov

Centre of Integrable Systems of YarSU, Yaroslavl, Russia;
mikhlchirkov@gmail.com

The main topic of this talk is n-simplex equations. They are generalization of famous Yang-Baxter and Zamolodchikov equations [6, 7], which have applications in many fields of mathematics and physics, including statistical mechanics, quantum field theory and integrable systems.

There are some methods connecting n-simplex equations with integrable systems [1, 3-5]. Thus, constructing new non-trivial solutions to n-simplex equations may lead to new integrable systems.

Let $\mathcal{X}$ be a set. Map

$$S : (x, y, z, t) \mapsto (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t), r(x, y, z, t))$$

is called *4-simplex map* if it satifsfies *4-simplex equation*

$$S^{1234} \circ S^{1567} \circ S^{2589} \circ S^{368,10} \circ S^{479,10} = S^{479,10} \circ S^{368,10} \circ S^{2589} \circ S^{1567} \circ S^{1234}.$$

Now, let $\mathrm{L} = \mathrm{L}(x)$ be 3×3 matrix, which depends on variable $x \in \mathcal{X}$

$$\mathrm{L}(x) = \begin{pmatrix} a(x) & b(x) & c(x) \\ d(x) & e(x) & f(x) \\ k(x) & l(x) & m(x) \end{pmatrix}$$

. Let $\mathrm{L}^6_{ijk}(x)$, $i, j, k = 1, \ldots 6$, $i < j < k$, — 6×6 matrix extensions of $\mathrm{L}(x)$.

We call this matrix equation

$$\mathrm{L}^6_{123}(u)\mathrm{L}^6_{145}(v)\mathrm{L}^6_{246}(w)\mathrm{L}^6_{356}(r) = \mathrm{L}^6_{356}(t)\mathrm{L}^6_{246}(z)\mathrm{L}^6_{145}(y)\mathrm{L}^6_{123}(x)$$

local tetrahedron equation. Solutions to this equation may generate solutions to 4-simplex equation.

In this talk we will present which matrices of this type

$$\mathrm{L}(x_{11}, \ldots, x_{33}) = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

may lead to new solutions of 4-simplex equations i.e. they have solution to local tetrahedron equation. For these matrices we construct new 4-simplex maps using methods from [2].

REFERENCES

1. *P. Kassotakis, M. Nieszporski, V. Papageorgiou, and A. Tongas.* Tetrahedron maps and symmetries of three dimensional integrable discrete equations. *J. Math. Phys.* **60** (2019) 123503
2. *S. Konstantinou-Rizos.* Birational solutions to the set-theoretical 4-simplex equation. Physica D: Nonlinear Phenomena **448** (2023), 133696
3. *S. Konstantinou-Rizos and A.V. Mikhailov.* Darboux transformations, finite reduction groups and related Yang–Baxter maps. *J. Phys. A: Math. Theor.* **46** (2013) 425201
4. *S. Konstantinou-Rizos and G. Papamikos.* Entwining Yang–Baxter maps related to NLS type equations. *J. Phys. A: Math. Theor.* **52** (2019) 485201
5. *V.G. Papageorgiou, A.G. Tongas, and A.P. Veselov.* Yang–Baxter maps and symmetries of integrable equations on quad-graphs. *J. Math. Phys.* **47** (2006) 083502
6. *A.B. Zamolodchikov.* Tetrahedra equations and integrable systems in three-dimensional space. *Sov. Phys. JETP* **52** (1980), 325–336.
7. *A.B. Zamolodchikov.* Tetrahedron equations and the relativistic S-matrix of straight strings in (2+1)-dimensions. *Commun. Math. Phys.* **79** (1981), 489–505.

ON THE HEATING CRISIS OF THE FLOW IN A PIPE

V. G. Danilov[1], **D. S. Minenkov**[2], **M. M. Votiakova**[3]

[1]*National Research University Higher School of Economics, Moscow, Russia; MIPT (National Research University), Dolgoprudny, Russia;*
vgdanilov@mail.ru

[2]*Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia; MIPT (National Research University), Dolgoprudny, Russia;*
minenkov.ds@gmail.com

[3]*Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia; MIPT (National Research University), Dolgoprudny, Russia;*
votiakova.mm@phystech.edu

The stationary motion of a liquid in a three-dimensional pipe with heating is considered. Based on Euler's equations, Darcy's law is valid for some modes and thus the problem is reduced to a one-dimensional system for density, pressure, temperature and longitudinal velocity (averaged in the cross-section). To model a liquid with supercritical pressure (as in [1]) we add van der Waals equation of state to close the system. As it is known (see [2]), setting boundary conditions are set for pressure lead to a solvability crisis: solution is only exists for small enough heating value and as heating approaches the critical value, temperature tends to infinity, density and velocity tend to zero. This can have an interpretation of overheating and blocking the stream. In [2] these effects were studied for a gas filtering with convection, here we reproduce the same effect for a supercritical fluid in a pipe. In the case, when values are set in the upcoming stream, which is more natural for a pipe problem, solution exists for any heating values. Nevertheless, even for small increment of heating large gradients of temperature and density occur, so that one can speak of heating crisis.

The authors are grateful to S.Yu. Dobrokhotov and A.A. Kovalishin for valuable discussions. The work was carried out in the frame of the government program (state registration No. 123021700044-0).

REFERENCES

1. *Alekseev G. V., Silin V. A., Smirnov A. M., Subbotin V. I.*, "Investigation of the temperature conditions of the pipe wall during heat removal by supercritical pressure water," High Temperature, **14**, No. 4, 769–774 (1976).
2. *Maslov V. P., Myasnikov V. P., Danilov V. G.*, Mathematical Modelling of the Chernobyl Reactor Accident, Moscow: Nauka (1988).

ASYMPTOTICALLY STABLE SOLUTIONS WITH BOUNDARY AND INTERNAL LAYERS IN DIRECT AND INVERSE PROBLEMS FOR A SINGULARLY PERTURBED HEAT EQUATION WITH A NONLINEAR THERMAL DIFFUSION COEFFICIENT

M. A. Davydova[1], **G. D. Rublev**[2]

[1]*Lomonosov Moscow State University, Faculty of Physics, Moscow, Russia;* m.davydova@physics.msu.ru
[2]*Obukhov Institute of Atmospheric Physics RAS, Moscow, Russia;* rublev.gd15@physics.msu.ru

In this paper, we propose a new approach to the study of direct and inverse problems for a singularly perturbed heat equation with a nonlinear thermal diffusion coefficient, based on the further development and use of asymptotic analysis methods in nonlinear singularly perturbed reaction-diffusion-advection problems [1-3]. The essence of the approach is presented on the example of a class of one-dimensional stationary problems with nonlinear boundary conditions, for which the case of applicability of asymptotic analysis is singled out. Sufficient conditions for the existence of classical solutions with boundary and internal layers for this class of problems are formulated, asymptotic approximations of an arbitrary order of accuracy of such solutions are constructed, algorithms for constructing formal asymptotics are substantiated, and the Lyapunov asymptotic stability of stationary solutions with boundary and internal layers as solutions of the corresponding parabolic problems is studied.

A class of one-dimensional nonlinear problems is considered, taking into account the lateral heat exchange with the environment according to Newton's law. An existence and uniqueness theorem for a classical solution with boundary layers in problems of this type is proved.

As an application of the results of study, the algorithms for solving specific direct and inverse problems of nonlinear heat transfer associated with an increase in the efficiency of operation of rectilinear heating elements in the melting furnaces-heat exchangers with convective or radiant modes are considered. Efficient numerical algorithms have been developed for solving inverse problems of restoring the thermophysical characteristics of nonlinear media (thermal conductivity and heat transfer coefficients), using a parameterized asymptotic approximation of the solution of direct problem or an equation that determines the position of the internal layer of solution of the direct problem.

REFERENCES

1. *Vasil'eva A. B., Butuzov V. F.,* Asymptotic decompositions of solutions of the singularly perturbed equations., Nauka, Moscow (1973).
2. *Vasil'eva A. B., Davydova M. A.,* "On a contrast steplike structure for a class of second-order nonlinear singularly perturbed equations", Comput. Math. Math. Phys., **6**, No. 38, 900–910 (1998).
3. *Nefedov N.N.,* "The method of differential inequalities for some singularly perturbed problems in partial derivatives", Differential Equations, **4**, No. 31, 718–722 (1995)

NOVEL SCENARIO FOR BRAESS'S PARADOX IN POWER GRIDS

A. S. Dmitrichev

Institute of Applied Physics of RAS, Nizhny Novgorod, Russia;
admitry@appl.sci-nnov.ru

The power grid is a complex network consisting of many interacting elements that provide the generation of electricity and transmit it to consumers [1]. Adding new elements (generators or consumers) in such a grid requires finding the optimal structure of the transmission lines connecting them to the grid, which ensures stable (synchronous) operation of the new grid. Typically, adding new transmission lines between grid elements improves the operation of the grid itself. However, there are unforeseen situations where the line addition violates the normal operation of the grid. This phenomenon is known as Braess's paradox [2]. To date, this paradox has been discovered in systems of very different nature (see references in [3]).

In the talk we consider several topologies of power grids and analyze how the addition of transmission lines affects their dynamics [3]. We establish conditions where the addition of the lines enhances stability of the grids or induces Braess's paradox reducing their stability. By using bifurcation theory and nonlocal stability analysis, we show that two scenarios for Braess's paradox are realized in the grids. The first scenario is well described and is associated with the disappearance of the synchronous mode. The second scenario has not been previously described and is associated with the reduction of nonlocal stability of the synchronous mode due to the appearance of asynchronous modes.

The work was supported by the Russian Science Foundation under Project No. 19-12-00338.

REFERENCES

1. *Sauer P. W., Pai A. M.* Power System Dynamics and Stabilityl, Englewood Cliffs: NJ: Prentice-Hall (1998).
2. *Braess D.* "*Ü*ber ein Paradoxon aus der Verkehrsplanung," Unternehmensforschung, **12**, 258–268 (1968).
3. *Khramenkov V. A., Dmitrichev A. S., Nekorkin V. I.* "A new scenario for Braess's paradox in power grids," Chaos, **32**, 113116-1–14 (2022).

DIRECT CHAOTIC WIRELESS COMMUNICATIONS

A. S. Dmitriev

Kotelnikov Institute of Radioengineering and Electronics RAS, Moscow, Russia; chaos@cplire.ru

The phenomenon of dynamic chaos (DC) has been discovered and actively studied since the mid-1960s. For the first 20 years, this phenomenon has been studied in detail with examples in various fields of natural science. Many properties of DC turned out to be surprising, and the possibility of synchronizing two or more systems with dynamic chaos certainly belongs to them. It was this property that caused the initial interest in chaos as a potential information carrier. And, although the first information transmission schemes proposed on the basis of chaotic synchronization turned out to be insufficiently effective in terms of noise immunity [1,2], the beginning of research on the use of dynamic chaos for information transmission was laid. In their process, it turned out that dynamic chaos has a number of properties that are attractive from the point of view of data transmission, namely: the possibility of obtaining complex oscillations using simple electronic devices; the ability to implement a large number of different chaotic modes in one device; control of chaotic modes by changing the system parameters; large information capacity; a variety of methods for inputting an information signal into a chaotic one; increasing the modulation rate in relation to the modulation of regular signals; the possibility of self-synchronization of the transmitter and receiver; non-traditional multiplexing methods; confidentiality in the transmission of messages.

This circumstance has stimulated new efforts to find methods and approaches for using chaotic signals in communications.

Among the potential candidates were relative transmission methods [2,3] and the use of energy reception [4,5]. Both approaches make it possible to use such features of chaotic signals as broadband (ultra-wideband), which, in accordance with Shannon's theory, provides a potential information capacity of the transmission channel proportional to the band and the possibility of using the spread of the information signal spectrum. At the same time, numerous studies in the field of using chaotic signals as an information carrier for communication systems show a significant gap between the potential information capabilities of communication systems based on dynamic chaos, listed above, and their practical feasibility.

Thus, the relative DCSK transmission scheme [2,3], popular in theoretical studies, for physical implementation requires delay lines with times of the order of the duration of the transmitted bits. In fact, for wireless communications, only a direct chaotic scheme with DCC energy reception is practically implemented [4,5].

The limited possibilities for the practical implementation of the existing theoretical communication schemes based on dynamic chaos hinders the development of chaotic communications and shows the need for new solutions that remove these limitations.

Another variant of the direct chaotic system is the recently proposed relative transmission scheme based on chaotic radio pulses. The report presents the results of studies of the characteristics and specifics of both schemes when they are used in ultra-wideband wireless communication in the centimeter and decimeter wavelength ranges.

REFERENCES

1. *Dmitriev A.S., Panas A.I.,* Dynamic Chaos. New Information Carriers for Communication Systems, Fizmatlit, Moscow (2002).
2. *Kaddoum G.,* "Wireless chaos based communication systems: a comprehensive survey," IEEE Access, 4, 2621–2648 (2016).
3. *Kolumban, G., Kennedy, M.P., Chua L.O.,* "The role of synchronization in digital communications using chaos. II. Chaotic modulation and chaotic synchronization," IEEE Trans., CS–45. No. 11, 1129–1140 (1998).
4. *Dmitriev A.S., Kyarginsky B.E., Panas A.I., Starkov S.O.,* "Direct-chaotic information transmission schemes in the microwave range," Raditaknika i elertronika, 46. No. 2, 224–233 (2001).

5. *Dmitriev A.S., Kyarginsky B. Ye., Panas A.I., Starkov S.O.*, "Experiments on ultra wideband direct chaotic information transmission in microwave band," Int. J. Bifurcation and Chaos, 13, No. 6, 1495–1507 (2003).

IDENTIFICATION OF TWO-POINT RAY PATHS REFLECTED ON A COASTLINE BY THE DIRECT VARIATIONAL APPROACH

S. Yu. Dobrokhotov[1], M. V. Klimenko[2], I. A. Nosikov[1], and A. A. Tolchennikov[3]

[1] *P. G. Demimdov Yaroslavl State University, Yaroslavl, Russia;* s.dobrokhotov@gmail.com; igor.nosikov@gmail.com
[2] *West Department of IZMIRAN, Kaliningrad, Russia;* maksim.klimenko@mail.ru
[3] *Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia;* tolchennikovaa@gmail.com

The direct variational method is a promising approach for solving the boundary problem of tsunami ray tracing. The main features of this approach are stability in the case of a large divergence of the rays and locality - only one ray is calculated connecting the start and end points. These circumstances make it possible to most effectively apply analytic-numerical formulas for the wave amplitude based on the Maslov operator. The main idea of the proposed variational approach is to successively optimize the ray path from some initial approximation to the desired optimum - the solution for which the objective function satisfies the stationarity principle. The presented method has the ability to determine both minima and saddle points of a given functional. A procedure for a systematic search for a set of stationary solutions based on the transitional properties of saddle points is implemented. In the framework of this work, the method is applied to calculate ray paths of tsunami waves, including in a medium specified from experimental bathymetry data. Implementations of the direct approach to the ray paths reflected from the coastline are considered. Verification was carried out with the results of traditional ray tracing procedure.

This work was supported by the Russian Science Foundation (project No. 21-71-30011).

ADAPTIVE MEASURE OF COHERENCE AS A UNIVERSAL ROBUST CRITERION FOR IDENTIFYING DYNAMIC STATES OF SPIKING NEURONAL NETWORKS

O. A. Dogonasheva[1,2], B. S. Gutkin[2], D. G. Zakharov[1]

[1]*Institute for cognitive neuroscience, National Research University Higher School of Economics, Moscow, Russia;* odogonasheva@hse.ru

[2]*Group of Neural Theory, École Normale Supérieure PSL* University, Paris, France;* boris.gutkin@ens.fr

[3]*Institute for cognitive neuroscience, National Research University Higher School of Economics, Moscow, Russia;* dgzakharov@hse.ru

Formation of synchronous activity patterns is an essential property of neuronal networks that has been of central interest to synchronization theory. Chimera states, where both synchronous and asynchronous activity of neurons co-exist in a single network, are a particularly poignant examples of such patterns, whose dynamics and multistability may underlie brain function, such as cognitive tasks. However, dynamical mechanisms of coherent state formation in spiking neuronal networks (SNN) as well as ways to control these states are remains unclear. In this paper we propose a robust universal approach to identify multiple network dynamical states, including stationary and travelling chimera states based on an adaptive coherence measure (ACM) [1]. Our approach allows automatic disambiguation of synchronized clusters, travelling waves, chimera states, and asynchronous regimes. We further couple our approach with a new speed calculation method for travelling chimeras [1,2].

Using this approach we studied the evolution of chimera states in a network of class II excitable Morris-Lecar neurons with asymmetrical nonlocal inhibitory connections [3]. We partitioned the network parameter space into regions of various collective behaviors (antiphase synchronous clusters, travelling waves, different types of chimera states as well as spiking death regime) and shown multistability between the various regimes (Fig. 1). Also we tracked the evolution of the chimera states as a function of key network parameters changed and found transitions between various types of chimera states.

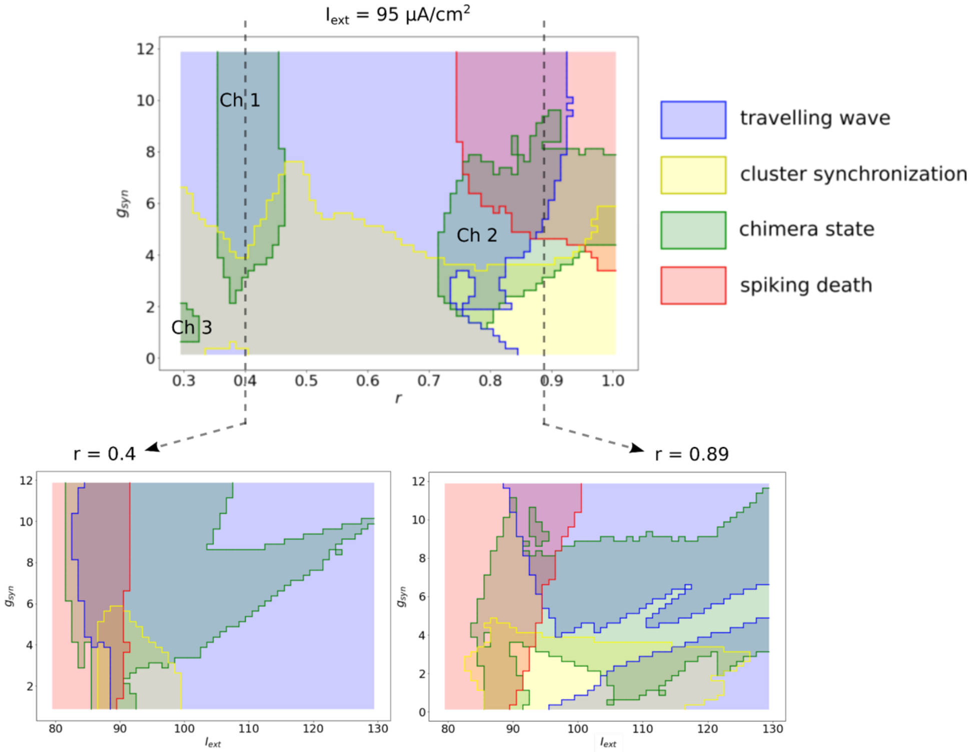

Fig. 1: Maps of the coherent states of the network of class II excitable Morris-Lecar neurons with asymmetrical nonlocal inhibitory connections. Parameters: I_{ext} is an neuronal external current, g_{syn} is a synaptic strength, and r is a connectivity parameter (the ratio of the number of non-local connections to the number of neurons) that describes connection density of the network. Regimes: travelling wave (blue), cluster synchronization (yellow), chimera state (green), spiking death (red). Due to multistability, the regions intersect and their colors mix: for example, the gray color in the lower left corner is the intersection of the regions of traveling waves (blue) and cluster synchronization (yellow). The borders of the regions keep their own colors. The modified figure from [3].

Our approach can be used not only for SNN, but it also works well for networks of various natures and, in particular, for the networks of phase oscillators [4]. In comparison with the other methods (order parameter [5], strength of incoherence [6] and χ^2-parameter [7]), the ACM-parameter is able to distinguish automatically between chimera states, cluster synchronization, as well as travelling waves and has not internal method parameters , which makes the calculations simple and robust.

The authors were supported by the Russian Science Fundation (project no. 23-

22-00418). This re- search was supported in part through computational resources of HPC facilities at NRU HSE.

REFERENCES

1. *Dogonasheva O. A., Kasatkin D. V., Gutkin B. S., Zakharov D. G.*, "Robust universal approach to identify travelling chimeras and synchronized clusters in spiking networks," Chaos, Solitons and Fractals, **153**, article 111541 (2021).
4. *Dogonasheva O. A., Gutkin B. S., Zakharov D. G.*, "Calculation of travelling chimera speeds for dynamical systems with ring topologies," in: Proceedings of the 5th Scientific School Dynamics of Complex Networks and their Applications (DCNA), IEEE, Kaliningrad, 2021, pp. 61–64.
3. *Zakharov D. G., Dogonasheva O. A., Kasatkin D. V., Gutkin B. S.*, "Multistability and evolution of chimera states in a network of type II Morris–Lecar neurons with asymmetrical nonlocal inhibitory connections," Chaos, **32**, article 101101 (2022).
4. *Dogonasheva O. A., Gutkin B. S., Zakharov D. G.*, "Bistability of globally synchronous and chimera states in a ring of phase oscillators coupled by a cosine kernel," in: Proceedings of the 5th Scientific School Dynamics of Complex Networks and their Applications (DCNA), IEEE, Kaliningrad, 2021, pp. 211–214.
5. *Abrams D. M., Strogatz S. H.*, "Chimera states for coupled oscillators," Physical review letters, **93**, article 174102 (2004).
6. *Gopal R., Chandrasekar V., Venkatesan A., Lakshmanan M.*, "Observation and characterization of chimera states in coupled dynamical systems with nonlocal coupling, " Physical Review E, 052914, 89 (2014).
7. *Golomb D., Hansel D., Mato G.*, "Mechanisms of synchrony of neural activity in large networks," in: Handbook of biological physics, volume 4, Elsevier, 2001, pp. 887–968.

THE THIRD TYPE OF CHAOS IN A SYSTEM OF TWO ADAPTIVELY COUPLED KURAMOTO OSCILLATORS

A. A. Emelianova

Institute of Applied Physics of the Russian Academy of Sciences, Nizhny Novgorod, Russia; emelianova@ipfran.ru

The report is devoted to the study of the phenomenon of mixed dynamics, which is considered the third type of deterministic chaos, characterized by the fundamental inseparability of conservative and dissipative behavior. In the system of adaptively coupled phase oscillators, the intersection of a chaotic attractor with a chaotic repeller was found, which indicates the existence of mixed dynamics in it. The system under consideration is the first general system (without symmetry with respect to time reversal) with a new type of chaos.

The author was supported by the Russian Science Foundation (project no. 19-12-00338).

REFERENCES

1. *Emelianova A. A., Nekorkin V.I.*, "On the intersection of a chaotic attractor and a chaotic repeller in the system of two adaptively coupled phase oscillators," Chaos, **29**, 111102 (2019).
2. *Emelianova A. A., Nekorkin V.I.*, "The third type of chaos in a system of two adaptively coupled phase oscillators," Chaos, **30**, 051105 (2020).
3. *Emelianova A. A., Nekorkin V.I.*, "Emergence and synchronization of a reversible core in a system of forced adaptively coupled Kuramoto oscillators," Chaos, **31**, 033102 (2021).
4. *Emelianova A. A., Nekorkin V.I.*, "The influence of nonisochronism on mixed dynamics in a system of two adaptively coupled rotators," Chaos, Solitons & Fractals, **169**, 113271 (2023).
5. *Shchapin D.S., Emelianova A. A., Nekorkin V.I.*, "A chaotic oscillation generator based on mixed dynamics of adaptively coupled Kuramoto oscillators," Chaos, Solitons & Fractals, **166**, 112989 (2023).

BIFURCATIONS OF PERIODIC REGIMES IN A MATHEMATICAL MODEL OF A DISCRETE RCL LINE WITH A TUNNEL DIODE

D. D. Fedulov

Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia; mr.fedulow@yandex.ru

Consider a mathematical model of a chain of N sequentially connected RCL-circuits, to the left end of which a DC power source is connected, and to the right end – a constant capacitance and a tunnel diode (see [1]):

$$\dot{u}_1 = -N(v_2 - v_1), \dot{v}_1 = -Nu_1 - \varepsilon v_1, \tag{1}$$

$$\dot{u}_n = -N(v_{n+1} - v_n), \dot{v}_n = -N(u_n - u_{n-1}) - \varepsilon v_n, \tag{2}$$
$$n = 2, \ldots N-1,$$

$$\dot{u}_N = -\frac{N}{1+\beta N}(\mu u_N - u_N^3 + v_N), \dot{v}_N = -N(u_N - u_{N-1}) - \varepsilon v_N. \tag{3}$$

Here ε, β, μ are positive parameters that are normalized parameters of the studied circuit, which satisfy the conditions:

$$0 < \varepsilon \ll 1, \quad \mu = \varepsilon\alpha, \quad \alpha = \text{const} > 0, \quad \beta = \text{const} > 0. \qquad (4)$$

Consider the problem of periodic solutions of the system (1)-(3). In the work [1], asymptotic methods were used to find orbitally stable cycles of the system (1)-(3). Changes in the parameter ε can lead to the loss of stability of such regimes.

In this work, bifurcations of periodic regimes for different values of N, α and β were studied when ε was increased. It has been shown that when ε is not small, a sufficiently large number of coexisting periodic regimes, tori, and chaotic attractors are preserved.

This work was supported by the Russian Science Foundation (project No. 22-11-00209), https://rscf.ru/en/project/22-11-00209/

REFERENCES

1. *Glyzin S. D., Kolesov A. Yu.*, "Self-oscillatory processes in a discrete *RCL*-line with a tunnel diode," Theoret. and Math. Phys., **215**, No. 2, 636–651 (2023).

ON POLAR FLOWS ON FOUR-DIMENSIONAL MANIFOLDS WITH TWO SADDLE EQUILIBRIA

D. O. Fomin

HSE University, Nizhny Novgorod, Russia; dofomin@edu.hse.ru

We recall that a smooth flow $f^t\colon M^n \to M^n$ defined on a closed smooth M^n manifold of dimension n is called a *polar flow* if:

1. its non-wandering set Ω_{f^t} consists of a finite number of hyperbolic equilibrium states, the set of source (sink) equilibrium states consists of a single point;
2. invariant manifolds of saddle equilibrium states intersect transversally.

A *Morse index* of a hyperbolic equilibrium state p is a number equal to the dimension of its unstable manifold W^u_p.

Let f^t be a polar flow on a manifold M^n and the set Ω_{f^t} consists exactly on a source α, a sink ω, and two saddles σ_i, σ_j of Morse indices i, j, correspondingly. For $n = 2, 3$ it is not hard to obtain an exhaustive list of manifolds admitting such flows: for $n = 2$ there are only torus and Klein Bottle, for $n = 3$ the manifold is a Lens space, including the sphere S^3 and the direct product $S^2 \times S^1$. For $n \geq 4$ the following theorem holds.

Theorem.

1. *if* $n \geq 4$ *and* $i = 1, j = n-1$ *then* M^n *is homeomorphic to* $S^{n-1} \times S^1$*;*
2. *case* $i = j = 1$ *or* $i = j = n-1$ *is impossible;*
3. *if* $n = 4$*,* $i = 2$*, and* $j = 1$ *or* $j = 3$ *then* M^4 *is homeomorphic to the sphere* S^4*.*

This theorem strengthens results of [1], [2], where similar polar flows were studied.

Research was supported by Program "Scientific Foundation of the National Research University Higher School of Economics" in 2023-2024 (project No 23-00-028).

REFERENCES

1. *Gurevich E. Ya., Pavlova D. A.* "O prostejshih potokah Morsa-Smejla s geteroklinicheskimi peresecheniyami na sfere S^n", Jhurnal SVMO, **19**, No. 2, 25–30 (2017).
2. *Zhuzhoma E. V., Medvedev V. S.,* "Underlying Manifolds of High-Dimensional Morse-Smale Diffeomorphisms with Two Saddle Periodic Points", Mathematical Notes, **109**, No. 4, 398–404 (2021).

PATCHED PATTERNS AND ONSET OF CHAOTIC INTERFACES: A NEW FORM OF SELF-ORGANIZATION IN NONLOCALLY COUPLED EXCITABLE SYSTEMS

I. Franović

Institute of Physics Belgrade, Belgrade, Serbia; `franovic@ipb.ac.rs`

While coherence-incoherence patterns have exhaustively been studied in systems of coupled oscillators, much less is known about the generic mechanisms of onset and the finite-size effects associated with such patterns in coupled excitable systems. Here we introduce a new class of patterns, called patched patterns [1], using a paradigmatic model of a non-locally coupled array of excitable FitzHugh-Nagumo units with attractive and repulsive interactions. Self-organization of such patterns involves the formation of spatially continuous domains, called patches, comprised of units locked by their average spiking frequencies. Typically, one distinguishes between the majority and minority patches with a 1:2 resonant frequency ratio. Patched patterns can be temporally periodic, quasiperiodic or chaotic, depending on the control parameter that modifies the prevalence of attraction vs repulsion. Chaotic patterns may further display interfaces, the regions comprising units whose frequencies are intermediate between the majority and minority patches.

Unlike chimeras, chaos in patched patterns is not spatially localized, but is manifested differently for the units within the patches and at the interfaces. The latter show greater variability and engage in chaotic itinerancy, reflected in switching between laminar epochs, where they appear transiently locked to the adjacent patches, and turbulent epochs of highly irregular activity. We demonstrate that the scenario of transition to chaos depends on the pattern wavenumber, affected by the coupling range. For smaller and intermediate coupling ranges, chaos typically emerges via torus breakup, while interfaces appear in a secondary bifurcation. Contrasting the classical result for chimeras, the maximal Lyapunov exponent of chaotic patched patterns does not decay, but instead converges to a finite value with the system size. Nevertheless, adapting the coupling range to reduce the pattern wavenumber to two changes the character of the transition to chaos. Then, the onset of chaos coincides with unpinning of interfaces, which show diffusive motion similar to that of the incoherent part of chimeras. In contrast to some other types of coherence-incoherence patterns in coupled excitable systems, such as solitary states, patched patterns turn out to be robust under noise [2].

REFERENCES

1. *Franović I., Eydam, S. R.*, "Patched Patterns and Emergence of Chaotic Interfaces in Arrays of Nonlocally Coupled Excitable Systems", Chaos, **32**, 091102 (2022).
2. *Franović I., Eydam, S. R., Semenova, N., Zakharova, A.*, "Unbalanced Clustering and Solitary States in Coupled Excitable Systems", Chaos, **32**, 011104 (2022).

LINEAR INTERFERENCE OF NONLINEAR WAVES IN MULTI-FIELD SYSTEMS

T. A. Gadzhimuradov[1], **A. M. Agalarov** [1,2]

[1]*Institute of Physics DFRC of RAS, Makhachkala, Russia;*
gta-1987@mail.ru

[2]*Dagestan State University, Makhachkala, Russia*

Among the great varieties of solitons, the vector ones that naturally arise in multi-field systems stand out, in particular [1]. It is remarkable that the requirement of elastic interaction can be weakened slightly for the vector class of solitons [2]. Vector solitons in nonlinear fiber optics are most thoroughly studied [3]. Even a single-mode fiber has birefringence, which requires the consideration of two orthogonally polarized components of field. The main obstacle for the formation of stable vector solitons here is the disagreement of group velocities for scalar envelope solitons, resulting from the difference in propagation constants for linearly polarized modes. A significant role in describing the dynamics of vector solitons is played by the Manakov model [4], which is the two component generalization of the nonlinear Schrödinger (NLS) equation [5].

Currently, a new type of "nondegenerate" bright vector solitons has been found within the Manakov focusing model [6]. A nondegeneracy is characterized by the presence of two distinct wave numbers in both modes (for one soliton solution); i.e., this term can be applied only to multi-field solitons. The list of suggested ones includes multispeed vector soliton (mss), which, in each component, contains solitons with absolutely arbitrary characteristics. The existence of such solitons in pure vector models is a nontrivial result and was predicted numerically earlier.

This paper argues that mss's are a linear interference of previously known degenerate solitons. This is confirmed numerically and analytically for the envelope solitons of anharmonically coupled nonlinear chains, which is given by the Hamiltonian

$$H = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left[\dot{u}_n^2 + \dot{v}_n^2 + W(u_n) + W(v_n) + \gamma u_n^2 v_n^2 \right], \tag{1}$$

where $W(u_n) = \alpha(u_n - u_{n+1})^2 + \beta u_n^4/2$ and $u_n(t)$, $v_n(t)$ measures the non-dimensional displacement with respect to their equilibrium positions. The numerical analysis for this model shows that the partial solitons propagate with constant and distinct velocities for the parametric choice $\gamma = 3\beta$. It is quite remarkable that when one spatial region is achieved (the cross-interaction term in Hamiltonian is significant), the capture and channeling of one-soliton component by another is not realized and there are no lower-order effects. So, the vector soliton is not originated and the pulses evolve as scalar solitons, but a small phase shift is present. This behavior can be analytically described. To describe this phenomenon by the combination of rotating wave approximation [7] and the Gardner–Morikawa transformation the equations of envelopes of partial solitons are obtained. The condition $\gamma = 3\beta$ turns out to be the condition for their integrability. Taking this into account and introducing the notation $\sigma = \gamma/2$, the following can get [8]

$$iU_\tau + U_{\xi\xi} + 2\sigma(|U|^2 + 2|V|^2)U + 2\sigma U^* V^2 = 0, \tag{2a}$$

$$iV_\tau + V_{\xi\xi} + 2\sigma(|V|^2 + 2|U|^2)V + 2\sigma V^* U^2 = 0. \tag{2b}$$

The searching for analytical solutions to eqs. (2) has been the goal of many investigations; it describe a single-mode fiber with low birefringence in the presence of coherent and incoherent interactions [3]. To obtain exact solutions, it can be noted that the linear transformation

$$U = \big(\Psi^{(1)} + \Psi^{(2)}\big)/2, \;\; V = \big(\Psi^{(1)} - \Psi^{(2)}\big)/2, \tag{3}$$

leads to a system of uncoupled NLS equations

$$i\Psi^{(\nu)}_\tau + \Psi^{(\nu)}_{\xi\xi} + 2\sigma\big|\Psi^{(\nu)}\big|^2\Psi^{(\nu)} = 0, \;\; \nu = 1, 2. \tag{4}$$

Transformation (3) causes interference for solitons of NLS eqs. (4). Taking into consideration the classical bright two-soliton solution of NLS equation (under certain conditions on the parameters), we can get that in $(U; V)$ solution, *the first component of the first vector soliton and the second component of the second soliton undergo a constructive interference, and the rest are subjected to the destructive one.* For clarity, two-mss is depicted in Fig. 1. It is also worth adding that such interference leads to a wide class of previously unknown solutions for the eqs. (2) and, as a consequence, for model (1).

So, in models (1) and (2), the mss structure is clear and understandable – it represents the interference of degenerate soliton solutions to eqs. (4). However, such interference mechanism is also valid

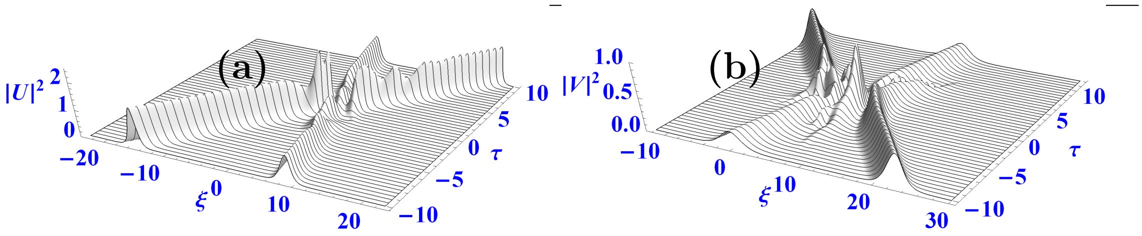

Fig. 2: [(a)-(b)] The multispeed two-soliton solution of eqs. (2). Each partial soliton has an unique velocity and trajectory.

for other multi-field systems. For instance, we consider classical n-coupled NLS equations (n-CNLS). Of course, a linear transformation similar to (3) does not split the system of n-CNLS equations into n independent NLS equations. In this case, it represents a mapping of the set of solutions onto itself, i.e. linear interference of any states will also be a solution [8, 9]. This is possible due to the fact that the n-CNLS equations has $\mathbb{U}(m + n)$ symmetry. Here also linear transformations generate a new range of exact solutions including mss. For example, in the case of the defocusing n-CNSE model, this makes it possible to construct dark solitons with a nontrivial carrier background wave [9].

REFERENCES

1. *Nayanov V. I.*, Multi-Field Solitons, FizMatLit, Moscow (2006).
2. *Agalarov A. M., Magomedmirzaev R.*, “Nontrivial class of composite $U(\sigma+\mu)$ vector solitons,” JETP Lett., **76**, No. 414–418 (2002).
3. *Kivshar Y. S., Agrawal G.*, Optical Solitons: From Fibers to Photonic Crystals, Academic press, San Diego (2003).
4. *Manakov S. V.* “On the theory of two-dimensional stationary self-focusing of electromagnetic waves,” Sov. Phys. JETP **38**, No. 248–253 (1974).
5. *Zakharov V. E., Shabat A. B.*, “Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media,” Sov. Phys. JETP **34**, No. 62 (1972).
6. *Stalin S, Ramakrishnan R., Senthilvelan M., Lakshmanan M.* “Nondegenerate solitons in Manakov system,” Phys. Rev. Lett. **122**, No. 043901 (2019).
7. *Arnold V. I., Kozlov V. V., Neishtadt A. I.*, Mathematical aspects of classical and celestial mechanics, Springer, Berlin (1997).
8. *Gadzhimuradov T. A.*, “Linear interference of nonlinear waves – Multispeed vector solitons,” Chaos **33**, No. 023101 (2023).
9. *Gadzhimuradov T. A.*, “Vector dark solitons with non-trivial periodic background,” Eur. Phys. J. Plus **138**, No. 279 (2023).

FOURIER TRANSFORM METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS WITH POWER NONLINEARITIES AND WITH CONSTANT COEFFICIENTS

V. I. Gishlarkaev[1], **A. A. Tovsultanov**[2]

[1]*Kadyrov Chechen State University, Grozny, Russia;*
vakhag@mail.ru, a.tovsultanov@mail.ru

This work is a continuation of articles [1-4], devoted to the application of the Fourier transform to the Cauchy problem for partial differential equations. In works [1-3], linear equations with variable (in the spatial variable) coefficients are considered. In the case of a special type of coefficients, equations with such coefficients are found in many applied problems, integral representations of solutions to the Cauchy problem are obtained. In a more general case, classes of functions are identified in which the existence and uniqueness of solutions is proven.

Using the approach from [1-3] in [4], problems for equations with power-law nonlinearities with certain conditions on the coefficients for these nonlinearities are considered. In this talk, we consider the case of equations with power-law nonlinearities with coefficients that are constant in spatial variables. Such coefficients are not subject to the above conditions. More precisely, we consider the Cauchy problem for equations of the form $\partial_t + Au = f(t,x)$, in which Au is the sum of differential operators with power nonlinearities $au^k(\partial_x^\alpha u)^l$ with coefficients $a = a_\alpha^{kl}(t)$.

The Fourier transform, which is a homomorphism from a convolutional multiplication algebra to a pointwise multiplication algebra, reduces the original equation to an integro-differential equation, which can be treated as an ordinary differential equation in a suitable Banach space. The resulting integral operator containing convolution powers instead of the differential operator A allows simple estimation in weighted L_2 norms, which greatly simplifies the analysis of the problem.

The main result of the work is the following theorem.

Theorem 5 *Let for some T_1 in the problem*

$$\partial_t u(t,x) + \sum_{|\alpha|\le M} a^{10}_\alpha(t)\partial^\alpha_x u(t,x) +$$
$$+\sum_{l=1}^{l_0} \sum_{l+k>1,k=0}^{k_0} \sum_{|\alpha|\le M} a^{lk}_\alpha(t) u^k(t,x)(\partial^\alpha_x u(t,x))^l =$$
$$= f(t,x),\ u|_{t=0} = u_0, \tag{1}$$

$f(\cdot,x) \in C([0,T_1])\ \forall x \in R^n,\ a^{lk}_\alpha(t) \in C([0,T_1]),\ u_0(x) \in C^A_F(R^n),$ $f(t,\cdot) \in C^A_F(R^n)\ \forall t \in [0,T_1]$

Then $\forall m_0 > M + n2^{-1} + p$, where p is some natural number depending only on l_0, k_0, M, $\exists T \le T_1 :\ \exists u \in L_1([0,T]; H^{m_0-p}(R^n))$ such that,$u(\cdot,x) \in C^1(0,t)\forall x \in R^n$, u is a classical solution to the problem (1)*.*

where the space of initial data and right-hand sides $C^A_F(R^n)$ is determined by the equality

$$C^A_F(R^n) = \{\Phi|_{R^n} : (\Phi : C^n \to C \text{is an entire function} :$$
$$((Im\Phi(x) = 0\ \forall x \in R^n) \wedge$$
$$(\textstyle\int_{R^n} |\Phi(x)|^2 dx < \infty) \wedge (\exists c, r \in R :\ |\Phi(z)| < ce^{r\|Imz\|_{R^n}}\ \forall z \in C^n))\},$$

where, in accordance with the Paley–Wiener theorems, the constant r is the radius of the ball containing the support $\check{\Phi}$, the value of c depends only on Φ.

The work of A. Tovsultanov was carried out within the framework of the state assignment of the Ministry of Education and Science of the Russian Federation under the project: Nonlinear singular integro-differential equations and boundary value problems (FEGS-2020-0001).

REFERENCES

1. *Gishlarkaev V. I.*, "Fourier transform method for partial differential equations: formulas for representing solutions to the Cauchy problem," Vestnik of Saint Petersburg University. Mathematics. Mechanics. Astronomy, **9**, No. 3, 480–494 (2022).
2. *Gishlarkaev V. I.*, "Fourier transform method for partial differential equations. Part 2. Existence and uniqueness of solutions to the Cauchy problem," Vestnik of Saint Petersburg University. Mathematics. Mechanics. Astronomy, **10**, No. 1, 21–35 (2023).

3. *Gishlarkaev V. I.*, “A method for representing solutions of the Cauchy problem for linear partial differential equations,” Sbornik: Mathematics, **209**, No. 2, 222–240 (2018).
4. *Gishlarkaev V. I.*, “Fourier transformation method for some types of nonlinear partial differential equations” Theoretical and Mathematical Physics, **207**, No. 3, 361–3750 (2021).

TRAVELING WAVE SOLUTIONS IN A RING NEURAL NETWORKS WITH A NEW MATHEMATICAL MODEL OF CHEMICAL CONNECTIONS

S. D. Glyzin[1], A. Yu. Kolesov[2]

[1]*Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia;* glyzin.s@gmail.com

[2]*Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia;* kolesov@uniyar.ac.ru

One of the fundamental principles for constructing mathematical models of neural systems is the equivalence hypothesis. The essence of this hypothesis is that we a priori assume the equivalence of a biological neuron to some electrical generator. This generator is modeled by a nonlinear system of ordinary differential equations or a system with delay. And since the oscillations of the membrane potential are obviously of a relaxation nature, the corresponding system turns out to be singularly perturbed.

In this paper, based on the equivalence hypothesis, to model the membrane potential $u = u(t) > 0$ of an individual neuron, we use a scalar nonlinear differential equation with delay of the form

$$\dot{u} = \lambda f(u(t-1))u. \tag{1}$$

Here the parameter $\lambda > 0$ characterizes the rate of electrical processes in the neuron. It is assumed to be large. Function $f(u) \in C^2(\mathbb{R}_+)$, $\mathbb{R}_+ = \{u \in \mathbb{R} : u \geq 0\}$, has the following properties: $f(0) = 1,\ f(u) + a = O\left(u^{-1}\right),\ \ uf'(u) = O\left(u^{-1}\right),\ \ u^2 f''(u) = O\left(u^{-1}\right)$ as $u \to +\infty$, where $a = \text{const} > 0$.

Let us now consider the problem of modeling chemical synapses and recall that a corresponding attempt was already made earlier in the article [1], where a certain approach to this problem was proposed, which was based on a modified idea of fast threshold modulation. Let us assume that a ring m, $m \geq 2$ of unidirectional chemically connected neurons is modeled. If we assume that each individual neuron is modeled by the equation (1), then based on the principle of fast threshold modulation, we can move from (1) to the system (see [1,2])

$$\begin{aligned} \dot{u}_j =&\lambda[f(u_j(t-1)) + b\,g(u_{j-1})h(u_j/u_{j-1})]u_j, \\ j =&1, \ldots, m,\ u_0 = u_m, \end{aligned} \tag{2}$$

where $b = \text{const} > 0$, $u_* = \exp(\sigma\,\lambda)$, $\sigma = \text{const} \in \mathbb{R}$, the functions $g(u), h(u) \in C^2(\mathbb{R}_+)$ are such that $g(u) > 0\ \forall\, u > 0$, $g(0) = 0$; $h(u) > 0$ for $0 \leq u < 1$, $h(u) < 0$ for $u > 1$, $h(1) = 0$, $h'(1) < 1$, $h(0) = 1$, $g(u) = 1 + O\left(u^{-1}\right)$, $ug'(u) = O\left(u^{-1}\right)$, $u^2 g''(u) = O\left(u^{-1}\right)$, $h(u) = -c + O\left(u^{-1}\right)$, $uh'(u) = O\left(u^{-1}\right)$, $u^2 h''(u) = O\left(u^{-1}\right)$ as $u \to +\infty$.

The resulting system is a new mathematical model of chemical synapses. An important property of this model is that after the replacements $x_j = \varepsilon \ln u_j$, $j = 1, 2, \ldots, m$, $\varepsilon = 1/\lambda \ll 1$ and ε tending to zero, has a relay system

$$\dot{x}_j = 1-(a+1)H(x_j(t-1))+b\,H(x_{j-1})[1-(s+1)H(x_j-x_{j-1})], \tag{3}$$

as a bounding object. Here $x_0 = x_m$,$1 \leq j \leq m$, and $H(x)$ is the Heaviside function.

The presence of the limit object (3) makes it easier to find the attractors of the (2) system and allows us to apply to it the general results from [3] on the correspondence between the rough cycles of the relay and relaxation systems. These results are used to find special periodic motions, the so-called traveling waves. A traveling wave is a special periodic solution of the system (3) that admits the representation $x_j = x(t+(j-1)\Delta)$, where $j = 1, 2, \ldots, m$, $k \in \mathbb{N}$, $1 \leq k \leq m-1$ is wave number, $\Delta = \text{const} > 0$ is some phase shift, and the function $x(t)$ is a periodic solution of an auxiliary equation

$$\dot{x} = 1-(a+1)H(x(t-1))+b\,H(x(t-\Delta))[1-(c+1)H(x-x(t-\Delta))].$$

We have shown that with an appropriate choice of parameters traveling waves of the relay system (3) can be found explicitly. In order to analyze the stability of these solutions, special numerical methods are used.

This work was supported by the Russian Science Foundation (project No. 22-11-00209), https://rscf.ru/en/project/22-11-00209/.

REFERENCES

1. *Glyzin S. D., Kolesov A. Yu., Rozov N. Kh.*, “On a method for mathematical modeling of chemical synapses,” Diff Equat., **49**, 1193–1210, (2013).
2. *Glyzin, S.D., Kolesov, A.Y.* “On a Method of Mathematical Modeling of Electrical Synapses,” Diff Equat, **58**, 853–868, (2022).
3. *Glyzin S. D., Kolesov A. Yu., Rozov N. Kh.*, “Self-excited relaxation oscillations in networks of impulse neurons,” Russian Math. Surveys, **70**, No 3, 383–452, (2015).

SHIFT MAP ON AN INFINITE-DIMENSIONAL TORUS

S. D. Glyzin[1], A. Yu. Kolesov[2]

[1]*Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia;* glyzin.s@gmail.com

[2]*Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Russia;* kolesov@uniyar.ac.ru

An infinite-dimensional torus $\mathbb{T}^\infty = \ell_p/2\pi\mathbb{Z}^\infty$ is considered (see, [1]). Here ℓ_p, $p \geq 1$ is a Banach space such that

$$\varphi = \text{colon}\,(\varphi_{(1)}, \ldots, \varphi_{(k)}, \ldots),\ \varphi_{(k)} \in \mathbb{R},\ k \geq 1,$$
$$||\varphi|| \stackrel{\text{def}}{=} \left(\sum_{k=1}^{\infty} |\varphi_{(k)}|^p\right)^{1/p} < \infty, \tag{1}$$

$\mathbb{Z}^\infty$ – integer lattice in ℓ_p of the form

$$\mathbb{Z}^\infty = \{l = \text{colon}\,(l_{(1)}, l_{(2)}, \ldots, l_{(k)}, \ldots) \in \ell_p :\ l_{(k)} \in \mathbb{Z},\ k \geq 1\} \tag{2}$$

We assume that two vectors $x, y \in \ell_p$ are equivalent if $x - y = 2\pi l$ for some $l \in \mathbb{Z}^\infty$. *The set of all equivalence classes generated by a given relation will be called a torus* $\mathbb{T}^\infty$. In particular, $\mathbb{T}^\infty = \ell_p/2\pi\mathbb{Z}^\infty = \text{pr}(\ell_p)$, where the map $\text{pr} : \ell_p \to \mathbb{T}^\infty$ is the so-called natural projection. This projection works according to the rule:

$$\text{pr} :\ \varphi \mapsto \{\varphi\}, \tag{3}$$

where φ is an arbitrary element from ℓ_p, and $\{\varphi\}$ is an equivalence class from $\mathbb{T}^\infty$, containing φ. Note that $\mathbb{T}^\infty$ is an Abelian group with respect to the addition operation defined by the rule:

$$\{\varphi_1\} + \{\varphi_2\} = \{\varphi_1 + \varphi_2\} \quad \forall\, \varphi_1, \varphi_2 \in \ell_p. \tag{4}$$

A shift map is a transformation of the form

$$G_\Delta : \varphi \mapsto \varphi + \Delta, \tag{5}$$

where φ *is an arbitrary point of the torus* $\mathbb{T}^\infty$, Δ *is a fixed element of* $\mathbb{T}^\infty$, *and the operation of addition is defined by rule* (4). The following equality is fulfilled

$$n\{\varphi\} = \{n\varphi\} \quad \forall\, \varphi \in \ell_p,\ \forall\, n \in \mathbb{Z}. \tag{6}$$

Thus, for any trajectory $\varphi_n = G_\Delta^n(\varphi_0)$, $\varphi_0 \in \mathbb{T}^\infty$, $n \in \mathbb{Z}$ of shift map (5) we have the equality

$$\varphi_n = \varphi_0 + n\Delta, \quad n \in \mathbb{Z}. \tag{7}$$

One of the problems of the modern theory of dynamical systems (see, for example, [2], [3]) is the question of the behavior of the trajectories of the standard shift map on the torus $\mathbb{T}^m, m \geq 2$. In this paper, this question is studied for the infinite-dimensional torus $\mathbb{T}^\infty$. As it turns out, the passage from $\mathbb{T}^m$ to $\mathbb{T}^\infty$ can lead to a new and paradoxical effect – the so-called turbulent behavior of trajectories. *This means that for some values of the parameters, both the* ω*-limit and* α*-limit sets of any trajectory of the shift operator on* $\mathbb{T}^\infty$ *are empty.* The connection of this effect with the problem of turbulence is discussed in the work [4].

This work was supported by the Russian Science Foundation (project No. 21-71-30011), https://rscf.ru/en/project/21-71-30011/

REFERENCES

1. *Glyzin S. D., Kolesov A. Yu.* "Elements of hyperbolic theory on an infinite-dimensional torus", Russian Math. Surveys, **77**, No. 3, 379–443, (2022).
2. *Katok A. B. and Hasselblatt B.* Introduction to the Modern Theory of Dynamical Systems, in Encyclopedia of Mathematics and its Applications (Cambridge Univ. Press, Cambridge, 1995), Vol. 54.
3. *Hasselblatt B. and Katok A. B.* A first course in dynamics with a panorama of recent developments, Cambridge Univ. Press, New York, 2003
4. *Kolesov A.Yu., Rozov N. Kh.* On the definition of 'chaos', Russian Math. Surveys, **64**, No. 4, 701–744, (2009).

ON DISCRETE LORENZ-LIKE ATTRACTORS IN THREE-DIMENSIONAL MAPS WITH THE AXIAL SYMMETRY

S. V. Gonchenko, **A. S. Gonchenko**

Mathematical Center of Lobachevsky State University, Nizhny Novgorod, Russia; sergey.gonchenko@mail.ru, agonchenko@mail.ru

One of the main goals of the paper is related to the construction of three-dimensional maps (diffeomorphisms of R^3) that would be both maximally simple and maximally similar in dynamics and bifurcations to three-dimensional Lorenz-like flows. First of all, such a map $(\bar{x}, \bar{y}, \bar{z}) = F(x, y, z)$ must have the *axial symmetry* $\{x \to -x, y \to -y, z \to z\}$, the same as in the Lorenz model, and must also have the *constant Jacobian* J, by analogy with the fact that the Lorenz system has the constant divergence. We shown that 3D axial symmetric maps with the constant Jacobian consist a specific class of 3D maps compliment to the well-known 3D Hénon-like maps. Among these axial symmetric maps, we select quadratic maps of the form $\bar{x} = y, \bar{y} = \nu_1 x + \nu_2 y + yz, \bar{z} = \nu_3 y + \alpha y^2$, where α is a coefficient equal to either -1 or $+1$ and ν_1, ν_2, ν_3 are parameters, and also $J = -\nu_1\nu_3$ is the Jacobian. Thus, for $\alpha = -1$ and $\alpha = +1$ we have two different maps that we denote T_- and T_+, respectively. We show that these maps possess the so-called dual symmetry which means that their second powers, maps T_-^2 and T_+^2, transform one to other under the change $y \to -y, z \to -z, \nu_2 \to -\nu_2, \alpha \to -\alpha$. This very useful property allows to simplify essentially the study of bifurcations of maps T_- and T_+. So, one can, for example, study all bifurcations of fixed points for both maps T_- and T_+, after which we will immediately know the bifurcations of their period-2 points. Despite this dual symmetry, as we show, maps T_- and T_+ demonstrate different bifurcations that involve into scenarios leading to the emergence of discrete Lorenz attractors. As we show, the latter are different in the case of maps T_- and T_+. Corresponding attractors A^- and A^+ contain both a symmetric saddle fixed point O. However, its unstable multiplier is positive for A^- and negative for A^+. Thus, the unstable separatrices Γ_1 and Γ_2 of O are invariant in the case of attractor A^-, i.e. $T_-(\Gamma_1) = \Gamma_1$ and $T_-(\Gamma_2) = \Gamma_2$, and, moreover, saddle fixed points resides in its two holes. This makes the attractor A^- to be remarkably similar to the classical Lorenz attractor. In contrast, Γ_1 and Γ_2 are semi-invariant in the case of attractor A^+, i.e. $T_+(\Gamma_1) = \Gamma_2$ and $T_+(\Gamma_2) = \Gamma_1$, and period-2 saddle orbit resides in its two holes. We also show that maps T_- and T_+ can have nonorientable discrete Lorenz-like attractors (when $J < 0$). We conduct also a comparative analysis of geometric structures of the attractors and demonstrate peculiarities of their homoclinic structures.

This paper was carried out in the framework of the RSciF under Grant No. 19-11-00280.

MULTIMODE DIFFUSION CHAOS IN LOGISTIC EQUATION WITH DELAY AND DIFFUSION ON A SEGMENT

V. E. Goryunov

Yaroslavl State University, Yaroslavl, Russia; salkar@ya.ru

We consider a diffusion chaos as a result of diffusion parameter decrease in a spatially distributed boundary value problem based on a logistic model with delay, which describes the dynamics of population density in a line segment:

$$\frac{\partial N}{\partial t} = D\Delta N + r\left(1 - N_{t-1}\right) N, \qquad \left.\frac{\partial N}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial N}{\partial x}\right|_{x=1} = 0.$$

Here $N(t, x)$ is the population density at time t and point x of a line segment Ω ($\Omega = \{x \mid 0 \le x \le 1\}$), Δ is a Laplace operator, D is a diffusion coefficient, r is a Malthusian coefficient of linear growth, $N_{t-1} \equiv N(t-1, x)$.

A problem of diffusion loss of stability of a spatially homogeneous mode was considered in [1, 2] in the case of a flat area. Analytical results were obtained at r close to critical value $\pi/2$ and small values of D, the key part of which can also be carried over to one-dimensional case. Also some numerical results at $r = 3$ were described, including a variety of complex modes. Stable modes at this value of r are not harmonic and are close to relaxation ones, this also applies to line segment case.

It is not possible to prove the well-known Oseledets theorem for considered type of systems, which allows us to calculate the Lyapunov exponents very efficiently. It means that we cannot calculate the Lyapunov dimension — a significant invariant characteristic of diffusion chaos (see [3, 4]). Nevertheless, a numerical algorithm was created that allows us to obtain the values close to Lyapunov exponents. This algorithm is described in paper [5]. It is based on the classical Benettin method with renormalization by the Gram–Schmidt algorithm and the fifth-order Dormand–Prince method (DOPRI54) with variable time step (see [6]) as a numerical method for solving the main system and the systems in variations. Thus, we can obtain some approximation of Lyapunov exponents spectrum and, thereby, we are able to apply the Lyapunov dimension formula.

Approximate values of Lyapunov dimension were calculated on 200 nodes partition of Ω at different values of D, at which diffusion chaos exists. It was found a sharp increase of the Lyapunov dimension with a decrease of diffusion parameter D.

This work was supported by the Russian Science Foundation (project No. 22-11-00209), https://rscf.ru/en/project/22-11-00209/

REFERENCES

1. *Glyzin S., Goryunov V., Kolesov A.*, "Spatially inhomogeneous modes of logistic differential equation with delay and small diffusion in a flat area," Lobachevskii Journal of Mathematics, **38**, No. 5, 898–905 (2017).
2. *Goryunov V. E.*, "Dynamics of solutions of logistic equation with delay and diffusion in a planar domain," Theor. Math. Phys., **212**, 1092–1110 (2022).
3. *Glyzin S. D., Kolesov A. Yu., Rozov N. Kh.*, "Finite-dimensional models of diffusion chaos," Comput. Math. and Math. Phys., **50**, 816–830 (2010).
4. *Glyzin S. D., Kolesov A. Yu., Rozov N. Kh.*, "Diffusion chaos and its invariant numerical characteristics," Theor. Math. Phys., **203**, 443–456 (2020).
5. *Goryunov V. E.*, "Features of the computational implementation of the algorithm for estimating the Lyapunov exponents of systems with delay," Modeling and Analysis of Information Systems, **26**, No. 4, 572–582 (2019).
6. *Hairer E., Nørsett S. P., Wanner G.*, Solving Ordinary Differential Equations I. Nonstiff Problems, Springer-Verlag Berlin Heidelberg (2008).

DYNAMICS OF LARGE SYSTEMS OF IDENTICAL OSCILLATORS WITH DELAYED COUPLINGS OF ADVECTIVE TYPES

E. V. Grigorieva[1], D. S. Kashchenko[2], S. A. Kashchenko[2,*], A. O. Tolbey[2,*]

[1]*Belarus State Economic University, Minsk, Republic of Belarus;*
[2]*Regional Scientific and Educational Mathematical Center of the Yaroslavl State University, Yaroslavl, Russia;*
grigorieva@tutby.news, d.kashchenko@uniyar.ac.ru, kasch@uniyar.ac.ru, a.tolbey@uniyar.ac.ru

We consider the local dynamics of chains consisting of a large number of identical elements. It is assumed that the connections are close to advective ones. They arise, for example, in standard difference approximations of the advection (transfer) operator. The transition from a system of many equations to a single integro-differential equation with a continuous spatial variable is carried out. Critical cases in the equilibrium stability problem are singled out and it is shown that they have infinite dimensionality. Another spatial variable arises when considering chains with delayed feedback under the assumption that the magnitude of the delay is sufficiently large.As the main results, special nonlinear boundary value problems of parabolic type — quasinormal forms are constructed, whose nonlocal dynamics describes the behavior of the principal terms of the asymptotic expansions of solutions from a small neighborhood of the equilibrium state of the original problem. The general components and differences in the dynamics of chains with different advective couplings are revealed.

We consider problems in which the coupling coefficients are strongly localized, i.e., when the chains are closest to the corresponding difference approximations. It is shown that in these cases additional infinite-dimensional critical cases appear. The corresponding quasinormal forms here are Ginzburg — Landau type equations with three spatial variables. As a general conclusion it is shown that the dynamics of the problems under consideration can be complex and irregular.

*The study was supported by the Russian Science Foundation, grant no. 21-71-30011, https://rscf.ru/en/project/21-71-30011/

REFLECTION EQUATION ALGEBRA AND RELATED COMBINATORICS

D. I. Gurevich

IITP, Moscow, Russia; dimitri.gurevich@gmail.com

Reflection Equation Algebras constitute a subclass of the so-called Quantum Matrix algebras. Each of the REA is associated with a quantum R-matrix. In a sense the REA corresponding to quantum R-matrix of Hecke type can be considered as q-counterparts of the commutative algebra Sym(gl(N)) or the enveloping algebras U(gl(N)). On any such RE algebra there exist analogs of some symmetric polynomials, namely the power sums and the Schur functions. In my talk I plan to exhibit q-versions of the Capelli formula, the Frobenius formula, related to this combinatorics. Also I plan to introduce analogs of the Casimir operators and partially perform their spectral analysis.

NEW OBSTRUCTION FOR MORSE–SMALE CASCADES TO EMBED IN TOPOLOGICAL FLOWS

E. Ya. Gurevich

NRU HSE, Nizhny Novgorod, Russia; egurevich@hse.ru

In report, an example of a Morse–Smale diffeomorphism given on $S^{n-1} \times S^1$, $n \geq 4$, that cannot be embedded in any topological flow, is provided. This diffeomorphism has no heteroclinic intersections, all its' periodic points are fixed, and closures of separatricies are trivial. This example contrasts to the fact, proven in [1], that all Morse–Smale cascade satisfying similar conditions and given on the sphere $S^n, n \geq 4$, do embed in topological flows, as well as similar Morse–Smale cascades on surfaces (that was proved in [2]). Necessary and sufficient conditions for Morse–Smale cascades given on manifolds of dimension greater than three to embed in a topological flows is discussed.

The author is partially supported by Laboratory of Dynamical Systems and Applications NRU HSE, grant of the Ministry of science and higher education of the RF, ag. No. 075-15-2022-1101.

REFERENCES

1. *Grines V., Gurevich E., Pochinka O.*, "On Embedding of Multidimensional Morse–Smale Diffeomorphisms into Topological Flows," Moscow Mathematical Journal, **19**, No. 4, 739–760 (2019).
2. *Palis J.*, "On Morse–Smale dynamical systems," Topology, **8**, 385–404 (1968).

KIRBY DIAGRAM AS A COMPLETE INVARIANT OF POLAR FLOWS ON 4-DIMENSIONAL MANIFOLDS

E. Y. Gurevich[1], **I. A. Saraev**[2]

[1]*HSE University, Nizhniy Novgorod, Russia;* egurevich@hse.ru
[2]*HSE University, Nizhniy Novgorod, Russia;* isaraev@hse.ru

Let M^n be a connected closed smooth manifold of dimension n. The flow f^t on M^n is called *polar* if it is structurally stable and its non-wandering set Ω_{f^t} consists of one source, one sink and finite number of saddle equilibrium states. We consider a class $\mathcal{P}(M^4)$ of polar flows on M^4 such that all saddle equilibria of any flow $f^t \in \mathcal{P}(M^4)$ have two-dimensional stable and unstable manifolds.

Let $f^t \in \mathcal{P}(M^4)$. It follows from [1],[2] that for the flow f^t there exists an *energy function* $\varphi : M^4 \to [0, 4]$, that is Morse function strictly decreasing along non-closed trajectories of f^t, having critical point at each equilibrium, and such that $\varphi(p) = \dim W^u_p$ for any $p \in \Omega_{f^t}$. Set $\Sigma_c = \varphi^{-1}(c)$, and $l_{p,c_1} = \Sigma_{c_1} \cap W^s_p$, $l_{p,c_2} = \Sigma_{c_2} \cap W^u_p$ for any saddle p and $c_1 \in (0, 2)$, $c_2 \in (2, 4)$. $\Sigma_{c_1}, \Sigma_{c_2}$ are smoothly embedded in M^4 spheres of dimension 3 and l_{p,c_1}, l_{p,c_2} are smooth simple closed curves (knots). Denote by $L_{c_1}(L_{c_2})$ the union of all such knots. Let $N_{p,c_2} \subset \Sigma_{c_2}$ be a closed tubular neighborhood of l_{p,c_2} and $\tilde{l}_{p,c_2} \in \partial N_{p,c_2}$ be a knot such that the linking number of $\tilde{l}_{p,c_2}, l_{p,c_2}$ equals zero.

Denote by $\eta_{c_1,c_2} : \Sigma_{c_2} \setminus L_{c_2} \to \Sigma_{c_1} \setminus L_{c_1}$ a diffeomorphism such that $\eta_{c_1,c_2}(x) = \mathcal{O}_{f^t}(x) \cap \Sigma_{c_1}$ for any point $x \in \Sigma_{c_2} \setminus L_{c_2}$, where $\mathcal{O}_x$ is the orbit of x. We will say that the knot $\tilde{l}_{p,c_1} = \eta_{c_1,c_2}(\tilde{l}_{p,c_2})$ is *framing* of the knot l_{p,c_1}.

We put $S_{f^t} = \Sigma_{c_1}$, $l_p = l_{p,c_1}$, $\tilde{l}_p = \tilde{l}_{p,c_1}$ and denote by L_{f^t} the set of all framed knots $\{l_p, \tilde{l}_p\}$. The pair S_{f^t}, L_{f^t} is called *Kirby diagram of the flow f^t*. This diagram is similar to diagram that was introduced by Kirby for classification of four-dimensional manifolds (see [3], [4]).

Theorem. *Flows $f^t, f'^t \in \mathcal{P}(M^4)$ are topologically equivalent if and only if there is exists homeomorphism $h : S_{f^t} \to S_{f'^t}$ such that $h(L_{f^t}) = L_{f'^t}$ and for any saddle equilibria p of the flow f^t there is exists the saddle equilibria p' of the flow f'^t such that $h(l_p) = l_{p'}$, $h(\tilde{l}_p) = \tilde{l}_{p'}$.*

Research was partially supported by Program "Scientific Foundation of the National Research University Higher School of Economics in 2023-2024 (project No 23-00-028) and by Laboratory of Dynamical Systems and Applications NRU HSE, grant of the Ministry of science and higher education of the RF, ag. No. 075-15-2022-1101.

REFERENCES

1. *Smale S.*, "On gradient dynamical systems," Annals of Mathematics, **74**, 199–206 (1961).
2. *Meyer K.*, "Energy Functions for Morse Smale Systems," Amer. J. Math., **90**, No. 4, 1031–1040 (1968).
3. *Kirby R.*, "A Calculus for Framed Links in S^3," Invent math., **45**, No. 1, 35–56 (1978).
4. *de Sa E.C.* "A link calculus for 4-manifolds," in: Fenn R. Topology of Low-Dimensional Manifolds, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 1979, pp. 16–31.

ON A METHOD FOR CLASSIFYING INTEGRABLE CHAINS WITH THREE INDEPENDENT VARIABLES

I. T. Habibullin[1], A. R. Khakimova[2], M. N. Kuznetsova[3]

[1]*Institute of Mathematics with Computing Centre - Subdivision of the Ufa Federal Research Centre of Russian Academy of Science, Ufa, Russia;* habibullinismagil@gmail.com

[2]*Institute of Mathematics with Computing Centre - Subdivision of the Ufa Federal Research Centre of Russian Academy of Science, Ufa, Russia;* aigul.khakimova@mail.ru

[3]*Institute of Mathematics with Computing Centre - Subdivision of the Ufa Federal Research Centre of Russian Academy of Science, Ufa, Russia;* mariya.n.kuznetsova@gmail.com

In the talk we will discuss a new classification method for nonlinear integrable chains with three independent variables using the chain class

$$u^j_{n+1,x} = u^j_{n,x} + f(u^{j+1}_n, u^j_n, u^j_{n+1}, u^{j-1}_{n+1}),$$

based on the use of reductions having the form of systems of differential–difference Darboux-integrable equations. The classification algorithm is based on the well-known fact that the characteristic algebras of Darboux-integrable systems have a finite dimension. Moreover, the structure of the characteristic algebra in the direction x for a given class of models is determined by some polynomial $P(\lambda)$, which can have one of the following four representations:

1. $P(\lambda) = \lambda^2 + a_1\lambda, \quad a_1 \neq 0,$
2. $P(\lambda) = \lambda^2,$
3. $P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda,$
4. $\deg P(\lambda) \geq 4.$

By now, we have made significant progress in cases 1) and 2), where, using the integrability conditions that follow from the finite-dimensionality of the characteristic algebra in the direction of x, we have presented an exhaustive list of candidates for integrability, containing several constant parameters, to find which we plan to use algebra in a discrete direction (see [1], [2]). Despite the fact that the classification problem is far from complete, we managed to find a new integrable equation of the form

$$u^j_{n+1,x} = u^j_{n,x} + (u^j_n - u^j_{n+1})(u^j_n + u^j_{n+1} - u^{j+1}_n - u^{j-1}_{n+1}),$$

for which a Lax pair is presented.

The study is supported by the Russian Science Foundation grant No. 21-11-00006, https://rscf.ru/en/project/21-11-00006/.

REFERENCES

1. *Kuznetsova M. N., Habibullin I. T., Khakimova A. R.*, "On the problem of classifying integrable chains with three independent variables," Theor. Math. Phys., **215**, No. 2, 667–690 (2023).
2. *Habibullin I. T., Khakimova A. R.*, "On the classification of nonlinear integrable three-dimensional chains by means of characteristic Lie algebras," arXiv:2306.14585 (2023).

INTEGRABLE SYSTEMS OF THE BOUSSINESQ TYPE AND THEIR FORMAL DIAGONALISATION

Heredero R. Hernández[1], **V. V. Sokolov**[2]

[1] *Universidad Polit ecnica de Madrid, Madrid, Spain;*
[2] *Kharkevich Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow, Russia;*
vsokolov1952@gmail.com

We discuss the integrability of Boussinesq type systems within the framework of the symmetry approach. A formal diagonalization of the linearization operator for such systems is proposed. This allows us to generate explicit integrability conditions in the form of canonical conservation laws. Using these conditions we perform a partial classification of systems. Most of the obtained equations are not polynomial.

GEOMETRY AND PROBABILITY ON THE NONCOMMUTATIVE 2-TORUS IN A MAGNETIC FIELD

M. N. Hounkonnou[1], **F. Melong**[2]

[1] *University of Abomey-Calavi, Cotonou, Benin Republic;*
norbert.hounkonnou@cipma.uac.bj
[2] *University of Abomey-Calavi, Cotonou, Benin Republic;*
fridomelong@gmail.com

This talk addresses the geometric and probabilistic properties of a noncomutative $2-$ torus in a magnetic field. We study the volume invariance, integrated scalar curvature and volume form by using the operator method of perturbation by inner derivation of the magnetic Laplacian operator in the noncommutative $2-$torus. Then, we analyze the magnetic stochastic process describing the motion of a particle subject to a uniform magnetic field on this manifold, and discuss the related main properties.

ALGEBRAIC CONSTRUCTIONS OF MIURA-TYPE TRANSFORMATIONS FOR DIFFERENTIAL-DIFFERENCE EQUATIONS AND ASSOCIATED NEW INTEGRABLE EQUATIONS

S. A. Igonin

Center of Integrable Systems, P.G. Demidov Yaroslavl State University, Yaroslavl, Russia; s-igonin@yandex.ru

It is well known that Miura-type transformations (MTs) play an essential role in the theory of integrable nonlinear partial differential and difference equations. (For partial differential equations, MTs are also called differential substitutions.)

In particular, when one tries to classify a certain class of integrable equations, one often finds a few basic equations such that all other equations from the considered class can be obtained from the basic ones by means of MTs (see, e.g., [7, 10, 4, 5, 6] and references therein). Applications of MTs to constructions of conservation laws [8, 9] and auto-Bäcklund transformations are also well known.

Therefore, it is highly desirable to develop systematic methods for constructing MTs. In this talk, we focus on MTs of differential-difference equations.

Extending results of [2] and adding some new ingredients, we present a method to construct MTs for differential-difference equations, using gauge transformations and invariants of group actions associated with Lax pairs of such equations. As a by-product, we show how one can simplify a Lax pair by gauge transformations in many cases.

The method is applicable to a wide class of Lax pairs. The considered examples include the (modified) Volterra, Narita–Itoh–Bogoyavlensky, Toda equations, a differential-difference discretization of the Sawada–Kotera equation, and Adler–Postnikov equations from [1]. Applying the method to these examples, we obtain new integrable nonlinear differential-difference equations connected with these equations by MTs.

Some steps of our method generalize (in the differential-difference setting) a result of V. G. Drinfeld and V. V. Sokolov [3] on MTs for the partial differential KdV equation.

If time permits, we will outline also an analogous method to construct MTs for difference-difference equations.

REFERENCES

1. *Adler V. E., Postnikov V. V.,* "Differential-difference equations associated with the fractional Lax operators," J. Phys. A: Math. Theor., **44**, 415203 (2011).
2. *Berkeley G., Igonin S.,* "Miura-type transformations for lattice equations and Lie group actions associated with Darboux–Lax representations," J. Phys. A: Math. Theor., **49**, 275201 (2016).
3. *Drinfeld V. G., Sokolov V. V.,* "On equations that are related to the Korteweg–de Vries equation," Soviet Math. Dokl., **32**, 361–365 (1985).
4. *Garifullin R. N., Yamilov R. I., Levi D.,* "Classification of five-point differential-difference equations," J. Phys. A: Math. Theor., **50**, 125201 (2017).
5. *Grammaticos B., Ramani A., Scimiterna C., Willox R.,* "Miura transformations and the various guises of integrable lattice equations," J. Phys. A: Math. Theor., **44**, 152004 (2011).
6. *Meshkov A. G., Balakhnev M. Ju.,* "Two-field integrable evolutionary systems of the third order and their differential substitutions," SIGMA, **4**, Paper 018 (2008).
7. *Mikhailov A. V., Shabat A. B., Sokolov V. V.,* "The symmetry approach to classification of integrable equations," in: What is integrability?, edited by V. E. Zakharov, Springer–Verlag, 1991, pp. 115–184.
8. *Miura R. M., Gardner C. S., Kruskal M. D.,* "Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion," J. Math. Phys., **9**, 1204–1209 (1968).
9. *Suris Yu. B.,* The Problem of Integrable Discretization: Hamiltonian Approach, Progress in Mathematics, 219, Birkhäuser Verlag, Basel (2003).
10. *Yamilov R.,* "Symmetries as integrability criteria for differential difference equations," J. Phys. A: Math. Gen., **39**, R541–R623 (2006).

COMMUTATIVITY OF SECOND-ORDER QUASIDERIVATIONS IN GENERAL LINEAR LIE ALGEBRAS

Y. Ikeda

Lomonosov Moscow State University, Moscow, Russia;
yasushikeda@yahoo.com

In this talk I speak about the quantum analogue of the theorem of A. Mishchenko and A. Fomenko, which allows one to construct commutative subalgebras in the universal enveloping algebras. We replace the derivation of the symmetric algebra $Sgl(d, \mathbb{C})$ (used in the original theorem) with the quasiderivation of the universal enveloping algebra $Ugl(d, \mathbb{C})$, proposed by Gurevich, Pyatov, and Saponov. In my paper I derived an explicit formula for such quasiderivations and used it to prove the quantum analogue of the theorem at the first order. In this talk I proceed with the calculations for the second order: we show that complex combinatorial formulas play an important role here.

REFERENCES

1. *Ikeda Y.*, "Quasidifferential operator and quantum argument shift method," Theoretical and Mathematical Physics, **212**, No. 1, 918–924 (2022).
2. *Mishchenko A., Fomenko A.*, "Euler equations on finite-dimensional Lie groups," Mathematics of the USSR-Izvestiya, **12**, No. 2, 371–389 (1978).
3. *Gurevich D., Pyatov P., Saponov P.*, "Braided Weyl algebras and differential calculus on $U(u(2))$," Journal of Geometry and Physics, **62**, 1175–1188 (2012).

STABILITY LOSS OF THE ZERO SOLUTION IN ONE SYSTEM OF DIFFUSIONALLY CONNECTED DIFFERENTIAL EQUATIONS WITH THE ADDITIONAL INTERNAL CONNECTION

L. I. Ivanovsky

Regional Scientific and Educational Mathematical Center of the Yaroslavl State University, Yaroslavl, Russia;
leon19unknown@gmail.com

Let us consider one system of differential equations with diffusion interaction between its neighboring elements and the additional internal connection

$$\dot{u}_j = N^2(u_{j+1} - 2u_j + u_{j-1}) + \gamma u_j - u_j^3, \qquad j = \overline{1, N}, \tag{1}$$

$$u_0 = u_1,\ u_{N+1} = u_N + \frac{\alpha}{N} u_k, \quad k \in \mathbb{N},\ 1 \leqslant k < N, \tag{2}$$

where functions $u_j(t)$ are smooth for $t \geqslant 0$ and parameters $\alpha, \gamma \in \mathbb{R}$. For homogeneous zero solution $u_j(t) \equiv 0$ of system (1), (2) there are two ways of stability loss: divergent, when zero appears in the stability spectrum of zero balance state, and oscillating, when a pair of complex conjugate eigenvalues have moved from the left complex half plane to the imaginary axis. Our research task was to find critical values of parameters α, γ and develop asympotic formulae for regimes derived from the zero balance state.

The obtained analytical results were illustrated by the numerical solution of system (1), (2) for the values of parameters close to the bifurcational ones. In addition, for values of parameter α close to critical ones, the normal form was constructed. It allowed us to determine conditions for the appearance of inhomogeneous balance states or cycles near the zero solution of system (1), (2).

This work was supported by the Russian Science Foundation (project No. 22-11-00209), https://rscf.ru/en/project/22-11-00209/.

REFERENCES

1. *Glyzin S. D.* "Difference Approximations of a Reaction–Diffusion Equation on Segments", Automatic Control and Computer Sciences, **52**, No. 7, 762–776 (2018).
2. *Kashchenko S. A.* "About Bifurcations at Small Perturbations in a Logistic Equation with Delay", Modeling and Analysis of Information Systems, **24**, No. 2, 168–185 (2017).
3. *Ivanovskiy L. I.* "Dynamics of One System of Diffusionally Coupled Differential Equations with Auxiliary Internal Connection", University proceedings. Volga region. Physical and mathematical sciences. Mathematics, **55**, No. 3, 15–30 (2020) (In Russian).

COMPLEX SYNCHRONOUS BEHAVIOR IN A NETWORK OF KURAMOTO OSCILLATORS WITH HETEROGENEOUS ADAPTIVE INTERACTION

D. V. Kasatkin

A.V. Gaponov-Grekhov Institute of Applied Physics of the Russian Academy of Sciences, Nizhny Novgorod, Russia;
kasatkin@appl.sci-nnov.ru

In the report, we discuss the features of synchronous behavior in a two-population network of Kuramoto oscillators with heterogeneous adaptive interactions. We analyze the influence of various heterogeneous interaction factors, such as the rules of coupling adaptation and the rate of their change, on the formation of various types of coherent network behavior. We demonstrate that such organization of interactions can lead to both suppression of some types of synchronous states observed in homogeneous networks [1] and the emergence of new modes such as transient clusters.

We found that various heterogeneous adaptation schemes lead to the formation of transient phase clusters of various types. In particular, the introduction of heterogeneity according to the coupling adaptation rule leads to the loss of stability of splay clusters and the emergence of transient synchronous behavior in the form of the circulant clusters [2]. In such states, each population contains a pair of anti-phase clusters whose size and composition slowly change over time as result of successive transitions of oscillators between clusters.

Another type of transient synchronous behavior, called pulsating clusters [3], was discovered when implementing the corresponding scheme of heterogeneity in the rate of coupling adaptation. This behavior could be observed in the case when the couplings between the oscillators of different populations changed faster than when within individual populations. The mode of pulsating clusters represents a sequential alternation of stages of synchronous and asynchronous network behavior.

The implementation of heterogeneity both in type and in the rate of change of adaptive couplings retains the possibility of the existence of transient clusters observed in the presence of only one of the heterogeneity factors and also leads to the appearance of transient synchronous states that have features of both circulant and pulsating clusters. This type of transient behavior is a cyclic sequential alternation of several stages of the evolution of the synchronous behavior of the network and a short interval of complete desynchronization of the oscillators. The stages of synchronous behavior include circulant rearrangements of clusters in populations, i.e., transitions of oscillators between clusters occurring in a certain direction and sequence.

The work was supported by the Russian Science Foundation (Project No. 23-42-00038).

REFERENCES

1. *Kasatkin D. V., Yanchuk S., Shöll E., Nekorkin V. I.,* “Self-organized emergence of multilayer structure and chimera states in dynamical networks with adaptive couplings,” Physical Review E, **96**, No. 6, 062211 (2017).
2. *Kasatkin D. V., Nekorkin V. I.,* “Transient circulant clusters in two-population network of Kuramoto oscillators with different rules of coupling adaptation,” Chaos, **31**, No. 7, 073112 (2021).
3. *Kasatkin D. V., Nekorkin V. I.,* “Transient phase clusters in a two-population network of Kuramoto oscillators with heterogeneous adaptive interaction,” Entropy, **25**, No. 6, 913 (2023).

DEPENDENCE OF THE NONLOCAL DYNAMICS OF THE COUPLED OSCILLATORS MODEL ON THE COUPLING TYPE

A. A. Kashchenko

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
a.kashchenko@uniyar.ac.ru

We study the nonlocal dynamics of the model of the ring of N coupled oscillators

$$\begin{gathered}\dot{u}_j + u_j = \lambda F(u_j(t-T)) + \gamma G(u_{j-1}, u_j, u_{j+1}), \quad j = 1, \ldots, N, \\ u_0 \equiv u_N, \quad u_{N+1} \equiv u_1.\end{gathered} \tag{1}$$

Here u_j ($j = 1, \ldots, N$) are real functions, the parameter T is positive, the positive parameter λ is large enough, γ is nonzero, F is a nonlinear compactly supported function (i. e. $F(x) \equiv 0$ if $x \in (-\infty, -p) \cup (p, +\infty)$), and G is a coupling function.

We study four coupling functions:

1. $G(u_{j-1}, u_j, u_{j+1}) = u_{j+1} - u_j$,
2. $G(u_{j-1}, u_j, u_{j+1}) = u_{j+1} + u_{j-1}$,
3. $G(u_{j-1}, u_j, u_{j+1}) = u_{j+1} - u_{j-1}$,
4. $G(u_{j-1}, u_j, u_{j+1}) = u_{j+1}$.

For each type of coupling we find conditions on γ for dissipativity of the model (1). We construct asymptotics of the model's (1) solutions with initial conditions from a wide subset of the phase space $C_{[-T,0]}(\mathbb{R}^N)$ on an interval $t \in [0, +\infty)$ and find conditions on the parameter γ for full synchronization and two-cluster synchronization in the model (1). We show that the range of parameter γ for dissipativity of the model (1) and asymptotics of its solutions depends on the function G crucially.

This study was supported by the President of Russian Federation grant (project no. MK-2510.2022.1.1).

PARABOLIC PDE ON TWO-DIMENSIONAL DOMAIN AS A NORMAL FORM OF SINGULARLY PERTURBED DDE

I. S. Kashchenko

P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;
iliyask@uniyar.ac.ru

It is known that parabolic PDE can be normal (quasinormal) form of differential equation with large delay [1, 2]. In this work we discuss more complex situations, when quasinormal form is parabolic PDE on two-dimensional domain.

For instance, it may be in the case of DDE with two large delays [3].

Consider the equation with two delays

$$\dot{x}+x = ax(t-T_1)+bx(t-T_2)+f(x, x(t-T_1), x(t-T_2)), \quad T_1 > T_2 > 0,$$

where $f(x, y, z)$ is nonlinear function ($f(0,0,0) = 0$ and each partial derivative $f'(0,0,0) = 0$). Main assumption is that both T_1 and T_2 are asymptotically large and $T_1 T_2^{-1}$ is large too. Let $T_1 = \varepsilon^{-1}$, where $0 < \varepsilon \ll 1$. Then $T_2 = \frac{1}{c\varepsilon^{1+\alpha}}$ ($\alpha > 0$). Let discuss the behavior of solutions in some small (but independed on ε) neighbourhood of zero equilibrium state. In critical cases we'll construct the analog of normal form – quasinormal form.

We proof that if $|a|+|b| < 1$ then $x = 0$ is stable and if $|a|+|b| > 1$ then zero is unstable. So $|a| + |b| = 1$ is critical case.

In the critical case there are eigenvalues of corresponding linear operator that located arbitrary close to the imaginary axis for small enough ε. These eigenvalues form a two-parameter countable family.

This dependence on two parameters leads to the fact that the constructed normal form contains two spatial variables [3].

This work was supported by the Russian Science Foundation (project No. 21-71-30011), https://rscf.ru/en/project/21-71-30011.

REFERENCES

1. *Kaschenko S. A.* "Application of the normalization method to the study of the dynamics of a differential-difference equation with a small factor multiplying the derivative," Differentsialnye Uravneniya **25**, 1448–1451 (1989).

2. *Kaschenko I. S.* “Application of the normalization method to the study of the dynamics of a differential-difference equation with a small factor multiplying the derivative “Local dynamics of equations with large delay,” Computational Mathematics and Mathematical Physics **48**, 2172–2181 (2008).
3. *Kaschenko I. S.* “Local dynamics of an equation with two large different-order delays,” Doklady Mathematics **94**, 578–582 (2016)

ASYMPTOTICS OF SELF-OSCILLATIONS IN CHAINS OF SYSTEMS OF NONLINEAR EQUATIONS

S. A. Kashchenko

Regional Scientific and Educational Mathematical Center of the Yaroslavl State University, Yaroslavl, Russia;
kasch@uniyar.ac.ru

Local dynamics of the chains of the coupled nonlinear systems of the second-order ordinary differential equations is studied. The basic assumption is that the number of elements of chains is large enough. This condition allows to pass on to the problem with a continuous spatial variable. The critical cases are considered while studying the stability of the equilibrum state. It is shown that they have infinite dimension. The research technique is based on the development and application of the special methods of the normal forms construction. The nonlinear boundary value problems of the parabolic type are construcred as the basic results. Their nonlocal dynamics describes the local behavior of solutions to the original system.

The study was supported by the Russian Science Foundation, grant № 21-71-30011, https://rscf.ru/en/project/21-71-30011/

THE INFLUENCE OF BOUNDARY CONDITIONS ON THE DYNAMIC PROPERTIES OF THE LOGISTIC EQUATION WITH DELAY AND DIFFUSION

S. A. Kashchenko[1,*], **D. O. Loginov**[1,*]

[1]*Regional Scientific and Educational Mathematical Center of the Yaroslavl State University, Yaroslavl, Russia;* kasch@uniyar.ac.ru, dimonl@inbox.ru

This work deals with the logistic equation with delay and diffusion which is essential in mathematical ecology. It is expected that the boundary conditions on each endpoint of the segment [0,1] contain parameters. The local dynamics in the neighborhood of the equilibrium state of the corresponding boundary value problem at all values of the parameters of the boundary conditions is investigated. The critical cases in the stability of the equilibrium state are found, and normal forms, which are scalar complex first-order ordinary differential equations, are constructed. Their non-local dynamics determines behaviour of original problem solutions in small neighborhood of equilibrium state. The problem of asymptotically small values of the diffusion coefficient in the dynamics of the boundary value problems under consideration is investigated separately. Specifically, it is shown that, in the construction of asymptotical solutions in the neighborhood of endpoints 0 and 1, boundary layer functions arise.

*The study was supported by the Russian Science Foundation, grant no. 21-71-30011, https://rscf.ru/en/project/21-71-30011/

MATRIX FACTORISATIONS AND PENTAGON MAPS

P. Kassotakis

University of Warsaw, Warsaw, Poland; pkassotakis@fuw.edu.pl

Set-theoretical solutions of the pentagon relation, also referred to as pentagon maps, turn to be essentially three-dimensional discrete integrable systems. In this talk we focus into a connection of the pentagon maps with discrete integrability, via Lax pair formulation. In detail, we propose a specific class of matrices that participate to factorization problems that turn to be equivalent to constant and entwining (non-constant) pentagon maps, expressed in totally non-commutative variables. We show that factorizations of order $N = 2$ matrices of this specific class, turn equivalent to the homogeneous normalization map, while from order $N = 3$ matrices we obtain an extension of the homogeneous normalization map, as well as novel entwining Pentagon, reverse-pentagon and Yang-Baxter maps.

REFERENCES

1. *Kassotakis, P.*, "Matrix factorizations and pentagon maps," arXiv:2302.02889 (2023).
2. *Kassotakis, P.*, "Non-Abelian hierarchies of compatible maps, associated integrable difference systems and Yang-Baxter maps," Nonlinearity, **36**, No. 5, 2514 (2023).

ON BLOW-UP AND GLOBAL EXISTENCE OF THE WEAK SOLUTIONS TO CAUCHY PROBLEM FOR A CERTAIN NONLINEAR EQUATION

I. K. Katasheva[1], **M. O. Korpusov**[2], **A. A. Panin**[3]

[1]*Moscow State University, Moscow, Russia;* kataschevа.ik15@physics.msu.ru

[2]*Moscow State University, Moscow, Russia;* korpusov@gmail.com

[3]*Moscow State University, Moscow, Russia;* a-panin@yandex.ru

The work concerns local and global solvability (in the sense of weak solutions, see [1]) and finite-time blow-up for the Cauchy problem for a model equation of the semiconductor theory:

$$\frac{\partial}{\partial t}\Delta u(x,t) - u(x,t) = |u(x,t)^q|, \quad x \in \mathbb{R}^3,$$

with initial condition $u(x,t) = u_0(x)$.

The method we use is the reduction to the integral equation in an appropriate weight space be means of the fundamental solution and of the potentials constructed with it. The fundamental solution has the form

$$\mathcal{E}(x,t) = -\frac{\theta(t)}{2\pi^{3/2}}\frac{1}{|x|}\int\limits_0^{+\infty} \exp\left(-\mu^2\right) J_0\left(2^{3/2}|x|^{1/2}t^{1/4}\sqrt{\mu}\right)\, d\mu,$$
$$x \in \mathbb{R}^3, \quad t > 0.$$

For $q > 4$, we estalish the existence of the non-extendable solution u, and if its domain is bounded then its norm tends to infinity as t tends to the end of the domain of u (we use the approach of [2]).

When adding the additional hypothesis $q > 8$ as well as hypothesis on the initial condition $u_0(x)$, we obtain that the domain of the solution is $t \geq 0$, that is, the solution is global. We show it using Gronwall–Bellman–Bihari inequality. On the other hand, for $q > 1$ the exist initial conditions for which the solution, if it exists, has only bounded domain. To show it, we use the test function method [3].

One of the authors (A. A. Panin) was supported by the Russian Science Foundation (project no. 23-11-00069).

REFERENCES

1. *Vladimiriv V. S.*, Equations of the mathematical physics [in Russian], Nauka, Moscow (1988).
2. *Panin A. A.*, “On local solvability and blow-up of solutions of an abstract nonlinear Volterra integral equation,” Journal, **97**, No. 6, 892–908 (2015).
3. *Pokhozhaev S. I., Mitidieri E.*, “A priori estimates and the absence of solutions of nonlinear equations and inequalites,” Trudy MIAN, **234**, 3–383 (2001) (in Russian).

THE SOLUTION OF THE BLACK BOX PROBLEM FOR ELECTRICAL NETWORKS VIA CLUSTER ALGEBRAS

A. A. Kazakov[1,2,3]

[1] *Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Russia;*
[2] *Centre of Integrable Systems, P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;*
[3] *Center of Fundamental Mathematics, Moscow Institute of Physics and Technology, Dolgoprudny, Russia;* anton.kazakov.4@mail.ru

Definition. A circular electrical network $e(G, \omega)$ is a planar graph embedding into a disk with a conductivity function $\omega : E(G) \to (\mathbb{R}_{\geq 0})^{|E(G)|}$.

Denote by $V_0(G)$ a set of all nodes lied on the disk boundary. Apply to $V_0(G)$ an electrical voltage $U_0 : V_0(G) \to (\mathbb{R}_{\geq 0})^{|V_0(G)|}$, which induces the voltage on nodes and electrical currents running through the edges of G. These voltages and electrical currents are defined by the Kirchhoff laws [2].

One of the main object in electrical network theory [2] is a response matrix $M_R(e)$: let us apply a voltage $\mathbf{U}$ to $V_0(G)$, it induces a currents $\mathbf{I}$ running through the boundary nodes. Values of $\mathbf{U}$ define $\mathbf{I}$ as follows:

$$M_R(e)\mathbf{U} = \mathbf{I}.$$

The cornerstone result related to $M_R(e)$ is the black box problem theorem [2]:

Theorem 1. *A conductivity function ω of any minimal circular electrical network $e(G, \omega)$ might be recovered by its response matrix $M_R(e)$.*

The last theorem might be proved via several constructive approaches, each of them is applied to finding numerical solutions of the famous Calder*ó*n problem [3] and, particularly, to the well-known inverse electrical impedance tomography problem [4].

My talk will be focused on the new approach to the black box problem theorem which is based on the cluster algebra for positive Grassmannians and the twist operation for them [5] .

This work was supported by the Russian Science Foundation (Grant No. 20–71–10110).

REFERENCES

1. *George, T.,* "Inverse problem for electrical networks via twist.," arXiv preprint arXiv:2305.10074, (2023).
2. *Curtis, E. B., Morrow, J. A.,* Inverse problems for electrical networks, World Scientific, (2000).
3. *Uhlmann, G.,* "Commentary on Calder*ó*n's paper (29), on an inverse boundary value problem," in: Selected papers of Alberto P. Caldero'n, pp 623–636, (2008).
4. *Borcea, L., Druskin, V., Vasquez, F. G.,* "Electrical impedance tomography with resistor networks," Inverse Problems, **24**, No. 3, pp. 035013, (2008).
5. *Muller, G., Speyer, D. E.,* "The twist for positroid varieties," Proceedings of the London Mathematical Society, **115**, No. 5, pp 1014–1071, (2017).
6. *Bychkov, B., Gorbounov, V., Kazakov, A., Talalaev, D.,* "Electrical networks, Lagrangian Grassmannians and symplectic groups," arXiv preprint arXiv:2109.13952, (2021).

PARTIAL STABILITY CRITERION FOR POWER GRIDS

V. A. Khramenkov

Institute of Applied Physics of RAS, Nizhny Novgorod, Russia;
hramenkov.vlad@appl.sci-nnov.ru

One of the priority tasks in studying power grids is to find the conditions of their stable operation [1]. The fundamental requirement is synchronizing all the elements (generators and consumers) of a power grid. However, various disturbances can destroy the synchronization. We consider the model of a heterogeneous power grid with hub structures (subgrids), taking into account arbitrary lengths and impedances of grid's transmission lines. Using the auxiliary comparison systems approach, we analyze the dynamics of a hub subgrid. Based on the findings, we develop a novel criterion of partial stability of power grids featuring hub structures [2]. The criterion makes it possible to identify the regions of safe operation of individual hub subgrid elements. First, the criterion allows obtaining parameters' values that guarantee existence and local stability of either synchronous or quasi-synchronous modes in individual hub subgrid elements, i.e. steady-state stability. Second, the criterion allows obtaining the safe values of abrupt frequency and phase disturbances that eventually vanish, i.e. imply transient stability with respect to state disturbance. Third, the criterion permits determining the safe ranges of parameters' disturbances that do not lead to catastrophic desynchronizing effects, i.e. imply transient stability with respect to parameters' disturbance.

The work was supported by the Russian Science Foundation under Project No. 19-12-00338.

REFERENCES

1. *Sauer P. W., Pai A. M.* "Power system dynamics and stability", Englewood Cliffs: NJ: Prentice-Hall (1998).
2. *Khramenkov V. A., Dmitrichev A. S., Nekorkin V. I.* "Partial stability criterion for a heterogeneous power grid with hub structures", Chaos, Solitons & Fractals, **152**, 111373 (2021).

MULTI-DIMENSIONAL BILLIARD BOOKS AND CLASSIFICATION PROBLEM

V. A. Kibkalo

Lomonosov Moscow State University, Moscow, Russia;
slava.kibkalo@gmail.com

Recent proof of several versions of Birkhoff conjecture [1, 2] shows that integrability of billiards in two dimensional domains is closely related to families of confocal quadrics.

From the topological point of view, i.e. the study of Liouville foliations of such systems, the class of billiards on CW-complexes with permutations (billiard books discovered by V.Vedyushkina [3]) is wide enough to model an arbitrary nondegenerate Morse-Bott singularity and an arbitrary base of Liouville foliation with such singularities (Fomenko invariant).

Construction of billiard books can be generalized to a multi-dimensional case. Having a family of confocal quadrics in R^n, one can consider a compact n-dimensional domain bounded by such quadrics as an n-dimensional cell. Each $n-1$-dimensional cell e^{n-1} is equipped with a cyclic permutation $(e^n_1, \ldots, e^n_k$ on the set of all n-dimensional cells $e^n_i, i = 1, \ldots, k$ such that closure contains $e^{n-1} \subset \bar{e^n_i}$. Each cell e^{n-2} of dimension $n-2$ corresponds to intersection of two confocal quadrics and is equipped with a commutation condition of two permutations corresponding to them. Such commutation conditions correctly determine reflection of a billiard ball for cells of dimension $0, 1, \ldots, n-3$.

In the talk we will discuss such generalization of billiard book construction and the classification problem for the 3-dimensional case. Although the classification problem for billiard books is still an open problem even for dimension 2, an implementation of a procedure suggested by the author in [4] can simplify it to the problem of classification of five concrete subclasses of CW-complexes with fixed commutation conditions. A natural generalization of this approach to the 3-dimensional case will be discussed.

The author was supported by the Russian Science Foundation (project no. 22-71-10106).

REFERENCES

1. *Glutsyuk A. A.*, "On polynomially integrable Birkhoff billiards on surfaces of constant curvature," J. Eur. Math. Soc, **23**, No. 3, 994–1049 (2021).

2. *Kaloshin V., Sorrentino A.*, "On the local Birkhoff conjecture for convex billiards," Annals of Mathematics, **188**, No. 1, 315–380 (2018).
3. *Vedyushkina V. V., Kharcheva I. S.*, "Billiard books model all three-dimensional bifurcations of integrable Hamiltonian systems," Sb. Math., **209**, No. 12, 1680–1727 (2018).
4. *Fomenko A. T., Kibkalo V. A.*, "Topology of Liouville foliations of integrable billiards on table-complexes," European J. of Mathematics, **8**, 1392–1423 (2022).

MEAN-FIELD DYNAMICS OF NEURAL NETWORK WITH GAUSSIAN PARAMETER DISTRIBUTION

S. Yu. Kirillov

A.V. Gaponov-Grekhov Institute of Applied Physics of the Russian Academy of Sciences, Nizhny Novgorod, Russia;
skirillov@ipfran.ru

The development of mean-field approaches for modeling the dynamics of neural networks is one of the rapidly expanding areas of nonlinear science. Within the framework of such approaches, macroscopic mathematical models are constructed that reproduce the behavior of ensemble-averaged characteristics of network activity, such as the mean frequency and the average membrane potential [1]. These macroscopic models can be used to describe the collective dynamics of large heterogeneous neural networks, provided that the individual heterogeneity parameters are Lorentz distributed. However, in reality, the parameters can be distributed in a significantly different way. Here we propose a method which allows to obtain macroscopic mathematical models for populations of quadratic integrate and fire neurons with an arbitrary parameter distribution [2]. The method is based on a fractionally rational approximation of a given distribution, which allows us to analytically obtain a reduced model directly from the dynamics of a microscopic population. We demonstrate the effectiveness of the proposed method using the Gaussian distribution and show that for an accurate description of a population of thousands of neurons, just a few equations for complex variables are sufficient.

The research was supported by the RSF (grant No. 19-72-10114).

REFERENCES

1. *Montbrió E., Pazó D., Roxin A.* "Macroscopic Description for Networks of Spiking Neurons," Physical review X, **5**, No. 2, 021028 (2015).
2. *Klinshov V., Kirillov S., Nekorkin V.* "Reduction of the collective dynamics of neural populations with realistic forms of heterogeneity," Physical Review E, **103**, No. 4, L040302 (2021).

DYNAMICS OF OSCILLATORS WITH PULSE DELAYED COUPLING

V. V. Klinshov

A. V. Gaponov-Grekhov Institute of Applied Physics of the Russian Academy of Sciences, 46 Ul'yanov street, Nizhny Novgorod, 603950, Russia; `vladimir.klinshov@ipfran.ru`

The work is devoted to the study of the dynamics of ensembles of oscillators with pulse delayed coupling. A method is described for reducing the dynamics of a system with impulsive delayed connections to a high-dimensional point mapping. The described method was used to study the dynamics of one oscillator with pulse delayed feedback [1], two oscillators with pulse delayed coupling [2], and an ensemble of oscillators with global connections [3]. Switching of pulsed oscillatory systems between different attractors under the influence of various types of noise has also been studied [4].

The work is supported by the Russian Science Foundation (grant no. 19-72-10114).

REFERENCES

1. *Klinshov V.V., Lücken L., Shchapin D.S., Nekorkin V.I., Yanchuk S.* "Multistable jittering in oscillators with pulsatile delayed feedback," Physical Review Letters, **114**, No. 17, 178103 (2015).
2. *Klinshov V.V., Nekorkin V.I.* "Synchronization of time-delay coupled pulse oscillators," Chaos, Solitons and Fractals, **4**, No. 1-3, 98 (2011).
3. *Klinshov V.V., Lücken L., Yanchuk S.* "Desynchronization by phase slip patterns in networks of pulse-coupled oscillators with delays," The European Physical Journal Special Topics, **227**, No. 10-11, 1117 (2018).
4. *Klinshov V.V., Shchapin D.S., D'Huys O.* "Mode hopping in oscillating systems with stochastic delays," Physical Review Letters, **125**, No. 3, 034101 (2020).

PHASE MODEL OF SPIKE AND BIRST ACTIVITY OF A NEURON

D. E. Kolesnikov[1], **D. A. Zaytseva**[1], **G. V. Osipov**[1]

[1]*Lobachevsky State University, N. Novgorod, Russia;*
osipov@itmm.unn.ru

The main form of electrical activity in a neuron is the so-called action potential, which is an electrical impulse or spike. Another common type of electrical activity is the spontaneous or induced generation of bursts, which are clusters of action potentials. These bursts are characterized by alternating periods of activity and quiescence. Bursting activity can exhibit both regular and chaotic patterns of generation. Neuronal bursting plays an important role in information transmission, generation, and synchronization of rhythms in neural networks. In the paper, we propose a computationally simple approach for phenomenological modelling of burst activity in neurons – the phi-neuron equation:

$$\dot{\varphi} = \gamma - \sin(\varphi/n). \tag{1}$$

The problem of synchronous behaviour of neurons in the case of spike and burst activity is being considered.

We acknowledge the financial support from RSF (Grant no. 23-12-00180).

STABLE SOLUTIONS OF QUASI-NORMAL FORMS OF A SPATIALLY DISTRIBUTED BOUNDARY VALUE PROBLEM

D. S. Kosterin

P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;
kosterin.dim@mail.ru

We consider the system of equations

$$\frac{du}{dt} = (A_0 + \varepsilon A_1)u + F_2(u,u) + F_3(u,u,u) + \varepsilon D_0 \frac{1}{2\pi}\int\limits_0^{2\pi} u(t,x)dx \tag{1}$$

where $u = u(t, x)$, $t \geq 0$, $x \in [0, 2\pi]$, A_0, A_1, D_0 are $n \times n$ matrices, $F_2(*, *)$, $F_3(*, *, *)$ are linear functions of their arguments, ε is small positive number.

This system is considered with periodic boundary condition

$$u(t, x + 2\pi) \equiv u(t, x). \tag{2}$$

We study the dynamics of the boundary value problem (1)–(2) near zero solution. If all the eigenvalues of the matrix A_0 have a negative real part, then the zero solution of the boundary value problem (1)–(2) is asymptotically stable.

In this work we consider the two critical cases in the stability problem. In the first critical case, matrix A_0 has zero eigenvalue. Then there is a one-dimensional manifold near zero solution of the boundary value problem, in which the dynamics of solutions is described by the equation

$$\frac{\partial \xi}{\partial \tau} = \xi - \xi^3 + \frac{\gamma}{2\pi} \int\limits_0^{2\pi} \xi(\tau, x) dx,$$

with periodic boundary condition

$$\xi(\tau, x + 2\pi) \equiv \xi(\tau, x).$$

This boundary value problem is called quasi-normal form.

In the second critical case, the matrix has two imaginary eigenvalues. In this case, there is a two-dimensional manifold near the zero solution of the boundary value problem, in which the dynamics of solutions is described by the equation

$$\frac{\partial \xi}{\partial \tau} = (1 + i\lambda_0)\xi + (-1 + i\sigma_0)\xi|\xi|^2 + \frac{\gamma}{2\pi} \int\limits_0^{2\pi} \xi(\tau, x) dx,$$

with boundary condition

$$\xi(\tau, x + 2\pi) \equiv \xi(\tau, x).$$

In both cases, there are piecewise smooth solutions of the quasi-normal form. These solutions have the form of step functions.

This work was supported by the Russian Science Foundation (project No. 22-11-00209), https://rscf.ru/en/project/22-11-00209/

REFERENCES

1. *Grigorieva E. V., Kashenko S. A.* “Slow and rapid oscillations in an optoelectronic oscillator model with delay,” Doklady Mathematics, **484**, No. 1, 21–25 (2019).
2. *Glyzin D. S., Glyzin S. D., Kolesov A. Yu.* “Hunt for chimeras in fully coupled networks of nonlinear oscillators,” Izvestiya VUZ. Applied Nonlinear Dynamic, **30**, No. 2, 152–175 (2022).

FORMAL STABILITY, STABILITY FOR MOST INITIAL CONDITIONS AND DIFFUSION IN ANALYTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

V. V. Kozlov

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia; vvkozlov@mi-ras.ru

An example of an analytic system of differential equations in $\mathbb{R}^6$ with an equilibrium formally stable and stable for most initial conditions is presented. By means of a divergent formal transformation this system is reduced to a Hamiltonian system with three degrees of freedom. Almost all its phase space is foliated by three-dimensional invariant tori carrying quasi-periodic trajectories. These tori do not fill all phase space. Though the “gap” between these tori has zero measure, this set is everywhere dense in $\mathbb{R}^6$ and unbounded phase trajectories are dense in this gap. In particular, the formally stable equilibrium is Lyapunov unstable. This behavior of phase trajectories is quite consistent with the diffusion in nearly integrable systems. The proofs are based on the Poincaré–Dulac theorem, the theory of almost periodic functions, and on some facts from the theory of inhomogeneous Diophantine approximations. Some open problems related to the example are presented.

The study was supported by the Russian Science Foundation, grant no. 21-71-30011, https://rscf.ru/en/project/21-71-30011/

NONLINEAR OSCILLATIONS OF A VERTICAL FLEXIBLE SHAFT WITH A SOLID DISK

E. P. Kubyshkin

P. G. Demidov State University, Yaroslavl, Russia;
kubysh.e@yandex.ru

Longitudinal-transverse vibrations of a vertical flexible shaft with a solid disk rotating at a constant angular velocity are investigated. It is assumed that the shaft and disc are homogeneous and perfectly balanced, the axes of the shaft and disc coincide, it is assumed that the shaft material is viscoelastic by nature, the ends of the shaft rest on bearings. The equations of longitudinal-transverse displacements of the points of the middle line of the shaft are obtained. It is shown that at certain rotational speeds, stable periodic solutions (precessions) and two-frequency solutions (beats) can occur in the equations. In the latter case, the shaft experiences longitudinal (vertical) vibrations, and the supports experience a strong shock load. The practical significance of the problem under consideration is related to the design and calculation of rotary systems. If the working angular speeds of rotation of the shaft (rotor) lie above some critical values, the question arises about the stability of rotation, as well as about the nature of the loss of stability. Loss of stability can lead to the emergence of both "soft" (most often asynchronous precession) and "hard" modes of occurrence of periodic oscillations, as well as to the emergence of modes of beats and chaotic oscillations. The last three options are the most dangerous. The problem under consideration also has important theoretical significance, since it simulates the dynamic behavior of shafts, which are the most important elements of many machines. An important component in constructing a model of the dynamics of distributed shafts is the model of internal friction of the material, since internal friction is a mechanism for transferring rotational energy into transverse and longitudinal vibrations. In the report, the shaft material is considered as subordinate to the nonlinear model of a hereditarily viscoelastic body. This leads to a new class of mathematical models of distributed rotating systems – nonlinear partial differential equations and infinite delay of the time argument. In this case, the boundary conditions contain the largest time derivatives and the time delay of the argument. Initial boundary value problems are formulated for this class of partial differential equations, solutions are determined and their solvability is shown. Such tasks have not been studied.

This work was carried out within the framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher Education of the Russian Federation (Agreement on provision of subsidy from the federal budget No. 075-02-2023-948).

STRUCTURALLY STABLE DEGENERATE SINGULARITIES OF INTEGRABLE SYSTEMS

E. A. Kudryavtseva[1], **M. V. Onufrienko**[2]

[1]*Moscow State University, Moscow, Russia; Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia;*
eakudr@mech.math.msu.su

[2]*Moscow State University, Moscow, Russia; Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia;*
mary.onufrienko@gmail.com

We obtain a classification (up to fiberwise diffeomorphism) of corank-1 singularities of typical integrable systems on 4- and 6-dimensional symplectic manifolds. We prove that, near a corank-1 orbit, the Liouville foliation of a typical integrable system is diffeomorphic to the standard model. We also depict phase portraits and their bifurcations near such singularities. Our singularities in the 4-dimensional case are parabolic orbits with resonances [2], and bifurcations of such orbits in the 6-dimensional case. Let us proceed with precise statements.

An integrable Hamiltonian system on a $2n$-dimensional symplectic manifold (M, ω) is given by a smooth mapping

$$\mathcal{F} = (f_1, \dots, f_n) : \ M \longrightarrow \mathbb{R}^n,$$

where $\{f_i, f_j\} = 0$. A *singular Lagrangian foliation* on (M, ω) arises whose fibers are connected components of the sets $\mathcal{F}^{-1}(c)$. Regular compact fibers are Liouville tori. Suppose that M is compact. Then the mapping $\mathcal{F}$ generates a Hamiltonian $\mathbb{R}^n$-action on M.

Definition. A local singularity (i.e., a compact $\mathbb{R}^n$–orbit) of an integrable system $(M, \omega, \mathcal{F})$ is called *structurally stable* if it has a neighborhood U in M such that any integrable system $(\overline{U}, \widetilde{\omega}, \widetilde{\mathcal{F}})$ sufficiently close to the given system in the C^∞-topology can be represented as follows: $\widetilde{\mathcal{F}} = \phi \circ \mathcal{F} \circ \Phi$, where $\Phi : \overline{U} \to M$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ are an embedding and a homeomorphism, resp., close to the identities. If Φ is a diffeomorphism onto its image, the singularity is called *smoothly structurally stable.*

Let $\mathcal{O}$ be a corank-1 local singularity, i.e., a compact $\mathbb{R}^n$-orbit such that $\operatorname{rk} d\mathcal{F}(\mathcal{O}) = n-1$. Then, in some neighborhood of $\mathcal{O}$, the system will be *semitoric* (if the system is real-analytic and the dimension of the fiber containing the orbit $\mathcal{O}$ is $\leq n$ [4]), i.e. will have the form

$$M_{st} = (D^2 \times D^p \times T^p)/G, \quad \omega_{st} = dx \wedge dy + dI \wedge d\varphi,$$

$$\mathcal{F}(x, y, I, \varphi) = \phi(H(x, y, I), I),$$

where

$$p = n-1,\ G = \langle A_s \rangle \subset SO(2),\ H = H(x, y, I) = H(x, y, I_1, \dots, I_p)$$

is a smooth G-invariant function of variables $(x, y) \in D^2$ and parameters $I = (I_1, \dots, I_p) \in D^p$, $\varphi = (\varphi_1, \dots, \varphi_p) \in T^p$, $s = |G| \in \mathbb{N}$, $A_s(x, y, I, \varphi) = ((x, y)A_s^\ell, I, \varphi_1 + \frac{2\pi}{s}, \varphi_2, \dots, \varphi_p)$, $0 < \ell < s$, $(\ell, s) = 1$, $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is diffeomorphism. Here $A_s = \begin{pmatrix} \cos\frac{2\pi}{s} & -\sin\frac{2\pi}{s} \\ \sin\frac{2\pi}{s} & \cos\frac{2\pi}{s} \end{pmatrix}$.

Definition. We define the *standard model* $(M_{st}, \omega_{st}, \mathcal{F}_{st})$ by setting

$$\mathcal{F}_{st} : M_{st} \to \mathbb{R}^n, \qquad \mathcal{F}_{st}(\mathbf{z}, I, \varphi) = (F_{s,k,\alpha(I)}(\mathbf{z}, I'), I),$$

where $0 \leq k < n$, $I' = (I_1, \dots, I_k)$, $\alpha(I)$ is a smooth function, and $F_{s,k,\alpha}(\mathbf{z}, \nu)$ is the function in two variables $\mathbf{z} = (x, y)$ and parameters $\nu = (\nu_i)_{i=1}^k$ defined below.

Thus, the standard model is the semi-toric integrable system $(M_{st}, \omega_{st}, \mathcal{F}_{st})$ with Hamiltonian $(\mathbb{R} \times T^{n-1})$-action, multipliers $e^{\pm\frac{2\pi\ell i}{s}} = \xi^{\pm\ell}$, twisting resonance ℓ/s [3] and resonance order $s = |G|$.

For any $s \in \mathbb{N}$, consider the group $G = \langle A_s \rangle \subset SO(2)$ of order s, see above, the G-invariant Morse functions $F_{s,0} = F_{s,0}^{\pm,\pm}(x, y) = \pm|z|^2 = \pm(x^2 + y^2)$ and $F_{s,0} = F_{s,0}^{+,-}(x, y) = x^2 - y^2$ (for $s = 1, 2$ only), and two families of $\mathbb{Z}_s$-invariant functions $F_{s,k,\alpha}(\mathbf{z},\nu)$, $k=1,2$:

$$F_{s,1,\alpha}(\mathbf{z},\nu)=\begin{cases} x^2+y^3 & +\nu y, \quad s=1,\\ \pm x^2+y^4 & +\nu y^2, \quad s=2,\\ \mathrm{Re}(z^3) & +\nu|z|^2, \quad s=3,\\ \mathrm{Re}(z^4)+\alpha|z|^4 & +\nu|z|^2, \quad s=4, \ \alpha^2\neq 1,\\ \mathrm{Re}(z^s)+|z|^4 & +\nu|z|^2, \quad s\geq 5,\end{cases}$$

$$F_{s,2,\alpha}(\mathbf{z},\nu)=\begin{cases} \pm x^2+y^4 & -\nu_2 y^2+\nu_1 y, \quad s=1,\\ \pm x^2+y^6 & +\nu_2 y^4+\nu_1 y^2, \quad s=2,\\ \mathrm{Re}(z^4)-|z|^4\pm|z|^6 & +\nu_2|z|^4+\nu_1|z|^2, \quad s=4,\\ \mathrm{Re}(z^5)+|z|^6 & +\nu_2|z|^4+\nu_1|z|^2, \quad s=5,\\ \mathrm{Re}(z^6)+\alpha|z|^6+|z|^8 & +\nu_2|z|^4+\nu_1|z|^2, \quad s=6, \ \alpha^2\neq 1,\\ \mathrm{Re}(z^s)\pm|z|^6+\alpha|z|^8 & +\nu_2|z|^4+\nu_1|z|^2, \quad s\geq 7.\end{cases}$$

Here $\nu=(\nu_i)\in\mathbb{R}^k$ is a small parameter, $\alpha\in\mathbb{R}$ is a parameter called *modulus*.

Consider the class $\mathcal{S}=\mathcal{S}(M)$ of integrable systems on M (called semitoric) for which the functions $f_2,\dots,f_n$ generate a locally free Hamiltonian action of the $(n-1)$-torus on M.

Theorem 1 ([5]). *Let $n=\frac{1}{2}\dim M\in\{2,3\}$ and $\mathcal{S}_{st}\subseteq\mathcal{S}$ denote the class of systems all of whose local singularities are fiberwise diffeomorphic to standard ones. Then $\mathcal{S}_{st}$ is open and dense in $\mathcal{S}$ with respect to the C^∞-topology. If the phase portrait of a standard model is preserved up to homeomorphism under a small change of the modulus α (for example, the modulus is absent), then any singularity fiberwise diffeomorphic to it is structurally stable (resp. smoothly structurally stable) with respect to small perturbations in the class $\mathcal{S}$.*

Remark. Theorem 1 describes parabolic trajectories with resonances [2] for $n=2$, and their typical bifurcations for $n=3$. Theorem 1 will be generalized to the case of n degrees of freedom if one finds a complete list (up to left-right equivalence) of G-invariant germs $f(x,y)$ having G-codimension $k\leq n-1$, their normal forms $F_{s,k,\alpha^*}(x,y,\mathbf{0})$ and their G-stable bifurcations $F_{s,k,\alpha}(x,y,\nu)$.

Theorem 1 does not follow from the classification [1].

The second author is a stipendiat of the Theoretical Physics and Mathematics Advancement Foundation “BASIS”.

REFERENCES

1. *Wassermann G.,* “Classification of singularities with compact Abelian symmetry,” Singularities Banach Center Publications, **20**, 475–498 (1988). PWN Polish Scientific Publishers, Warsaw (1988).
2. *Kalashnikov V. V.,* “Typical integrable Hamiltonian systems on a four-dimensional symplectic manifold,” Izv. Math., **62**, No. 2, 261–285 (1998).
3. *Kudryavtseva E. A.,* “Hidden toric symmetry and structural stability of singularities in integrable systems,” European J. Math., **8**, 1487–1549 (2022).
4. *Zung N.T.,* “A note on degenerate corank-one singularities of integrable Hamiltonian systems,” Comment. Math. Helv., **75**, 271–283 (2000).
5. *Kudryavtseva E. A., Onufrienko M. V.,* “Classification of singularities of smooth functions with a finite cyclic symmetry group,” Russian Journal of Mathematical Physics, **30**, No. 1, 76–94 (2023).

ON THE QUESTION OF THE UNIQUENESS OF THE CENTER MANIFOLD

A. N. Kulikov

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
anat_kulikov@mail.ru

Consider the system of autonomous ordinary differential equations

$$\dot{x} = Ax + F(x, y), \tag{1}$$

$$\dot{y} = By + G(x, y), \tag{2}$$

where $x = colon(x_1, \ldots, x_m), y = colon(y_1, \ldots, y_k), x_j = x_j(t), y_s = y_s(t), F(x, y) = colon(F_1, \ldots, F_m), G(x, y) = colon(G_1, \ldots, G_k), F_j = F_j(x_1, \ldots, x_m, y_1, \ldots, y_k), G_s = G_s(x_1, \ldots, x_m, y_1, \ldots, y_k), j = 1, \ldots, m, s = 1, \ldots, k$. We assume that the vector functions $F(x, y), G(x, y)$ depend sufficiently smoothly on their arguments and, moreover, $F(0, 0) = 0, G(0, 0) = 0$, and also at zero have an order of smallness higher than the first. Finally, A, B are square matrices of order m and k, respectively. Moreover, for all eigenvalues of the matrix A the equality $Re\lambda_j = 0$, and for the eigenvalues of the matrix B the inequalities $Re\lambda_s < 0$ is valid.

Then, as is well known [1-3], the system of differential equations (1), (2) has a locally invariant manifold

$$M_h : \, y = h(x), \tag{3}$$

where

$$h(x) = (h_1, \ldots, h_k), h_s = h_s(x_1, \ldots, x_m), s = 1, \ldots, k, h_s(x_1, \ldots, x_m) -$$

enough smooth functions having an order of smallness at zero above the first.

A fairly well-known result is that the central invariant manifold (3) (M_h) is not unique. Examples can be found in [2,3]. However, sufficient conditions can be given that guarantee the uniqueness of M_h.

Theorem. *Let the equation $y = g(x)$ define some central invariant manifold M_g (one of the manifolds M_h). Consider the system of differential equations*

$$\dot{x} = -Ax - F(x, g(x)) \tag{4}$$

and let the zero solution of system (4) be asymptotically stable. Then the center manifold M_g is unique, see [4], where this result was presented.

This work was carried out within the framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher Education of the Russian Federation (Agreement on provision of subsidy from the federal budget No. 075-02-2023-948).

REFERENCES

1. *Hartman F.* Ordinary differential equations, Mir, Moscow (1970) (in Russian).
2. *Pliss V. A.* “The principle of reduction in the theory of stability of motion,” Izv. AH USSR. Ser. mathematicheskaya, **28**, No. 6, 1297–1324 (1964) (in Russian).
3. *Kulikov A. N., Kulikov D. A., Sekatskaya A. V.* Invariant manifolds: some applications to the theory of dynamical systems, Filigran, Yaroslavl (2022) (in Russian).
4. *Kulikov A. N.* “On the uniqueness of invariant manifolds,” In sb.: Research on stability and oscillation theory, 81–85 (1979) (in Russian).

NONLOCAL EROSION EQUATION AND NANORELIEF FORMATION

D. A. Kulikov

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
kulikov_d_a@mail.ru

The formation of a spatially nonhomogeneous relief on the surface of semiconductor materials under the influence of ion bombardment is one of the problem for modern nanoelectronics. To describe this technological process, various mathematical models are used. As one of them, an equation was proposed, which was called the "nonlocal erosion equation" [1]

$$u_t = au_{xx} - cw_x + b(u-w) + b_1(u-w)w_x + c_1 w_x^2 + b_2(u-w)w_x^2 + c_2 w_x^3. \quad (1)$$

As a rule, equation (1) is supplemented with periodic boundary conditions

$$u(t, x + 2\pi) = u(t, x). \quad (2)$$

The boundary-value problem (1), (2) is given in a normalized form. The function $u = u(t,x)$ defines the relief shape, $w = u(t, x-h)$. Here $a, h > 0, c, b \geq 0$
$(c^2 + b^2 \neq 0), b_1, b_2, c_1, c_2 \in \mathbb{R}$.

Usually two options for choosing the coefficients of the equation are considered:

$$a)\, c = c_1 = c_2 = 0,\ b > 0\,(b = 1);\ \ b)\, b = b_1 = b_2 = 0,\ c > 0\,(c = 1).$$

For both options, the following questions were considered:

1) on the stability of homogeneous equilibrium states;

2) about local bifurcations when an equilibrium state changes its stability.

We add that from a physical point of view, as the main bifurcation parameter, one can naturally choose either a or h (sometimes in special cases both a and h at the same time).

Problem 1. Let $c = c_1 = c_2 = 0, b > 0\,(b = 1)$, and a is considered as the main parameter. With this version of the problem statement, it is shown that:

1) for $a > a_*\,(a_* > 0)$ homogeneous equilibrium states $u = const$ are stable and they are unstable if $a \in (0, a_*)$;

2) for $a = a_*$ critical case in the problem of stability of solutions $u = const$ is realized;

3) for $a = a_*(1 - \nu\varepsilon)\,(\nu = \pm 1, \varepsilon \in (0, \varepsilon_0))$ the question of local bifurcations is studied and sufficient conditions for the formation of a nonhomogeneous nanorelief are obtained .

We emphasize that the coefficient a is inversely proportional to the energy of the ion flux.

Problem 2. A similar problem arises if $b = b_1 = b_2 = 0, c > 0\,(c = 1)$ and h (deviation value) is chosen as the main bifurcation parameter. For this option we can show that:

1) if $h \in (0, h_*)\,(h_* > 0)$, then the equilibria $u = const$ are stable and they are unstable if $h \in (h_*, \infty)$;

2) for $h = h_*$ a critical case in the problem of their stability is realized;

3) for $h = h_*(1 + \gamma\varepsilon)\,(\gamma = \pm 1, \varepsilon \in (0, \varepsilon_0))$ sufficient conditions for the formation of a nonhomogeneous nanorelief are obtained.

The proof of the results is based on the methods of the theory of dynamical systems with an infinite-dimensional phase space. Some bifurcation problems for the nonlocal erosion equation were considered in [2-4].

REFERENCES

1. *Rudy A. S., Bachurin V. I.* “Spatially nonlocal model of surface erosion by ion bombardment,” Bulletin of the Russian Academy of Sciences: Physics, **72**, No. 5, 586–591 (2008).
2. *Kulikov A. N., Kulikov D. A.* “Nonlocal model of relief formation under the influence of an ion flux. Nonhomogeneous nanostructures,” Mat. modelirovnie, **28**, No. 3, 33–50 (2016) (in Russian).
3. *Rudyi A. S., Kulikov A. N., Kulikov D. A., Metlitskaya A. V.* “High-mode wave reliefs in a spatially nonlocal erosion model,” Russian Microelectronics, **43**, No. 4, 277–2283 (2014).
4. *Kulikov D. A.* “Spatially Ingomogeneous dissipative structures in a periodic boundary-value problem for a nonlocal erosion equation,” Journal of Mathematical Sciences, **205**, No. 6, 791–805 (2015).

THE TOPOLOGY OF HIGHER-ORDER COMPLEXES IN FUNCTIONAL BRAIN NETWORKS ASSOCIATED WITH MAJOR DEPRESSIVE DISORDER

S. A. Kurkin[1], **A. E. Hramov**[1]

[1]*Immanuel Kant Baltic Federal University, Kaliningrad, Russia;*
kurkinsa@gmail.com

The analysis of functional networks provides great opportunities for understanding the processes occurring in the human brain both in normal and pathological states [1]. For example, various psychiatric disorders are characterized by subtle deviations from the norm of functional connections at different network scales.

Traditionally, functional networks are analyzed only at the level of pairwise interactions. However, this simplification does not allow to study the simultaneous interactions of several brain areas, which are typical for complex brain mechanisms. The application of methods for analyzing higher-order interactions in networks is a more powerful and effective technique. One such approach is Q-analysis, which identifies strongly connected simplicial complexes in different topology levels q [2]. We apply this method to the brain functional networks reconstructed from resting-state fMRI data of healthy subjects and patients with major depressive disorder (MDD) to reveal higher-order network features associated with MDD. We calculated and analyzed the following characteristics: the first, second, and third structure vectors, the topological dimension, and the topological entropy.

To overcome the problem of high inter-subject variability, we utilize the concept of the so-called "consensus network" [3]. A consensus network is a characteristic network consisting of connections that are shared among most individuals within a given group.

We have found that MDD patients have fewer higher-order interactions between brain regions, and these connections are weaker compared to the control group. We have also revealed that functional networks in healthy subjects are characterized by a higher degree of modularity and clustering compared to MDD patients.

The present study highlights the importance of studying higher-order interactions in functional brain networks, especially when considering pathological conditions, including depression. The results indicate abnormalities in interregional interactions in MDD patients.

The authors were supported by the Russian Science Foundation (project no. 23-71-30010).

REFERENCES

1. *Hramov A. E., Frolov N. S., Maksimenko V. A., Kurkin S. A., Kazantsev V. B., Pisarchik A. N.*, "Functional networks of the brain: from connectivity restoration to dynamic integration," Physics-Uspekhi, **64**, No. 6, 584 (2021).
2. *Beaumont J. R., Gatrell A. C.*, An introduction to Q-analysis, Geo Abstracts, Norwich (1982).
3. *Andjelkovic M., Tadic B., Melnik R.*, "The topology of higher-order complexes associated with brain hubs in human connectomes," Scientific reports, **10**, No. 1, 17320 (2020).

THE INVESTIGATION OF DYNAMIC EVOLUTION OF THE COMPACT PLANETARY SYSTEM K2-72

E. D. Kuznetsov[1], **A. S. Perminov**[2]

[1] *Ural Federal University, Yekaterinburg, Russia;*
eduard.kuznetsov@urfu.ru
[2] *Ural Federal University, Yekaterinburg, Russia;*
alexander.perminov@urfu.ru

We consider the dynamic evolution of the compact four-planetary system K2-72 (http://exoplanet.eu/catalog/all_fields/). Star K2-72 is an M-type dwarf with a mass of $m_{star} = 0.27^{+0.08}_{-0.09}$ $M_\odot$. The orbital periods of the planets in days are $P_b = 5.577212 \pm 0.00042$, $P_d = 7.760178^{+0.001496}_{-0.001496}$, $P_c = 15.189034 \pm 0.0031$, and $P_e = 24.158868^{+0.003726}_{-0.00385}$. Estimates of the eccentricities of orbits for all planets are $0.11^{+0.02}_{-0.09}$. The inclinations of the orbits of the planets lie within 2 degrees.

The system contains three Earth-like planets K2-72 b, K2-72 d, K2-72 c with radii $R_b = 1.08\ R_\oplus$, $R_d = 1.01\ R_\oplus$, and $R_c = 1.16\ R_\oplus$. The K2-72 e planet is super-Earth with radius $R_e = 1.29\ R_\oplus$. We used the following relation [1] for the density of a planet $\varrho_p = 2.43 + 3.39(R_p/R_\oplus)$ on its radius R_p when $R_p < 1.5R_\oplus$ to estimate the masses of planets: $m_b = 4.18 \cdot 10^{-6}\ M_\odot$, $m_d = 3.28 \cdot 10^{-6}\ M_\odot$, $m_c = 5.41 \cdot 10^{-6}\ M_\odot$, and $m_e = 7.95 \cdot 10^{-6}\ M_\odot$. We searched for low-order resonances within the uncertainty of determining the periods of the planets. To determine the resonant combinations of the semi-major axes of the orbits, the ratio of the mean motions of neighboring planets was represented as a segment of a sequence of convergent fractions. We did not find resonances of the form $(p+q):p$ in the K2-72 system, for which $2 \le p+q \le 10$ at $1 \le q \le 5$.

For compact systems, an important factor affecting evolution is tidal interaction; therefore, we considered a number of scenarios for the evolution of the K2-72 system over 100 Myr using the Posidonius software [2], which takes into account tidal interactions. At nominal values of the periods and eccentricities, the simulation ended with the decay of the system 3 Myr after the start of integration. This is because at eccentricities of 0.11, the distance at the apocenter of the orbit of the planet K2-72 b becomes greater than the distance at the pericenter of the orbit of the planet K2-72 d. If the initial values of eccentricities of orbits of all planets are set equal to the minimum value of 0.02, the system remains stable over the entire considered interval of 100 Myr for all values of star masses from 0.18 to 0.35 $M_{\odot}$. The results of simulations including tidal interactions show that the compact four-planet system K2-72 can have a stable dynamical evolution in the absence of low-order resonances. Based on the results of numerical simulations with the Posidonius software, we plan to obtain limits on the maximum values of eccentricities of the orbits of the planets, at which the system remains stable over 100 Myr.

The authors were supported by the Russian Ministry of Science and Higher Education via the State Assignment Project FEUZ-2020-0038.

REFERENCES

1. *Weiss L. M., Marcy G. W.*, "The mass–radius relation for 65 exoplanets smaller than 4 Earth radii," The Astrophysical Journal Letters, **783**, L6 (2014).
2. *Blanco-Cuaresma S., Bolmont E.*, "Studying tidal effects in planetary systems with Posidonius. A N-body simulator written in Rust," in: EWASS Special Session 4 (2017): Star-planet interactions (EWASS-SS4-2017), EWASS, Prague, 2017, doi:10.5281/zenodo.1095095

PROPAGATION OF AN AUTOWAVE FRONT IN A DISCONTINUOUS MEDIUM

N. T. Levashova[1], **E. A. Chunzhuk**[2]

[1]*Department of Mathematics, Faculty of Physics, Lomonosov Moscow State University, 119991 Moscow, Russia;*
levashovant@physics.msu.ru

[2]*Department of Mathematics, Faculty of Physics, Lomonosov Moscow State University, 119991 Moscow, Russia;*
chunzhuk.ea18@physics.msu.ru

We study the solution of a moving front form to the initial boundary value problem

$$\begin{cases} \varepsilon^2 \dfrac{\partial^2 u}{\partial x^2} - \varepsilon \dfrac{\partial u}{\partial t} = (u - \varphi^{(-)}(x))(u - q(x))(u - \varphi^{(+)}(x)), & x \in (-1, 1), \\ \dfrac{\partial u}{\partial x}(-1, t, \varepsilon) = \dfrac{\partial u}{\partial x}(1, t, \varepsilon) = 0, \quad u(x, 0) = u_{init}, & t \in (0, T], \end{cases}$$

with discontinuous function

$$q(x) = \begin{cases} q_l(x), & -1 \le x \le x_0; \quad 0 < t < T; \qquad q_l(x_0) \ne q_r(x_0). \\ q_r(x), & \phantom{-1 \le{}} x_0 \le x \le 1; \quad 0 < t < T, \end{cases}$$

Here ε is a small parameter.

We consider the front at each moment t is localized in the vicinity of the point $\hat{x}(t)$. The initial function also is of front form with localization point $\hat{x}(0) < x_0$.

Let the inequalities hold:

Proposition 1

$$\frac{\varphi^{(-)}(x) + \varphi^{(+)}(x)}{2} < q_l(x) < \varphi^{(+)}(x), \quad -1 \le x \le x_0,$$

.

Proposition 2

$$\varphi^{(-)}(x_0) < q_l(x_0) < q_r(x_0).$$

Under these propositions, we proved the existence theorem and constructed the uniform asymptotic approximation in the small parameter exponents of the front form solution to problem under consideration. In the study, we used the Vasil'eva method [1] and the asymptotic method of differential inequalities [2].

The authors were supported by Russian Science Foundation (project no. 23-11-00069).

REFERENCES

1. *Vasil'eva, A. B., Butuzov, V. F., Kalachev, L. V.,* The Boundary Function Method for Singular Perturbation Problems, Society for Industrial and Applied Mathematics: Philadelphia (1995).
2. *Nefedov, N. N.,* "Development of methods of asymptotic analysis of transition layers in reaction-diffusion-advection equations: theory and applications" in the journal Comput. Math. Math. Phys., **61**, No. 12, 2068–2087 (2021).

EXACT SOLUTION OF THE PROBLEM ON FORCED VIBRATIONS OF A STRING OF VARIABLE LENGTH

V. L. Litvinov[1], **K. V. Litvinova**[2]

[1]*Samara State Technical University, Samara, Russia;*
vladlitvinov@rambler.ru
[2]*Moscow State University, Moscow, Russia;*
kristinalitvinova900@rambler.ru

The article considers the resonant characteristics of forced vibrations of a string with moving boundaries. The analytical method was developed in relation to obtaining an exact solution of the wave equation with a higher class of conditions on moving boundaries that differ from the boundary conditions of the first kind.

The differential equation describing forced strings is:

$$Z_u(x,t) - a^2 Z_{xx}(x,t) = \omega_0^2 B \cos W_0\big(\omega_0 t\big).$$

Border conditions:

$$Z(0,t) = 0; \quad Z\big(l_0(t), t\big) = 0.$$

The initial conditions do not affect the resonant properties of linear systems, so they are not considered in this problem [1]. Performing transformations, similar to transformations [2, 3], we obtain an expression for the total amplitude at the point $\xi = \xi_0(\tau)$, corresponding to the maximum amplitude of oscillations

$$A_n^2(\tau) = \left\{ \left[\int_0^{b(\tau)} F_n(\zeta) \cos \Phi_n(\zeta) d\zeta \right]^2 + \left[\int_0^{b(\tau)} F_n(\zeta) \sin \Phi_n(\zeta) d\zeta \right]^2 \right\}.$$

Thus, an expression for the amplitude of system oscillations in the nth dynamic mode has been obtained using the application of special functional equations.

REFERENCES

1. *Litvinov V. L.*, “Variational formulation of the problem of vibrations of a beam with a movable spring support,” Journal of Theoretical and Mathematical Physics, Russian Academy of Sciences, **215**, No. 2, 289–296 (2023).
2. *Litvinov V. L., Litvinova K. V.*, “Numerical scheme for solving problems that describe the longitudinal-transverse vibrations of objects with moving boundaries in the developed software package TB-Analisys,” International scientific journal ”Modern information technologies and IT education”, **18**, No. 3 (2023).
3. *Litvinov V. L., Litvinova K. V.*, “An Approximate Method for Solving Boundary Value Problems with Moving Boundaries by Reduction to Integro-Differential Equations,” Journal of Computational Mathematics and Mathematical Physics, **62**, No. 6, 977–986 (2022).

GENERALIZED MISHCHENKO–FOMENKO CONJECTURE FOR LIE ALGEBRAS OF SMALL DIMENSIONS

F. I. Lobzin

Lomonosov Moscow State University, Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia;
fiadat@mail.ru

Let $\mathfrak{g}$ be a Lie algebra, respectively, $\mathfrak{g}^*$ be the dual space. Consider on $\mathfrak{g}^*$ the structure:

$$\mathcal{A}_x(x) = (c_{ij}^k a_k),\ a, x \in \mathfrak{g}^*.$$

This tensor defines the Lie-Poisson bracket on $C^\infty(\mathfrak{g}^*)$. The functions $f \in C^\infty$, lying in the kernel of the Lie-Poisson bracket are called Casimir functions.
It is also possible to consider a similar structure which is called a bracket with a frozen argument:

$$\mathcal{A}_a(x) = (c_{ij}^k a_k),\ a, x \in \mathfrak{g}^*.$$

A Lie algebra is called completely integrable if over this Lie algebra there is a complete set of functions that are in involution. A complete set is considered to contain n functionally independent functions, where n is defined by the formula:

$$n = \frac{1}{2}(dim\ \mathfrak{g} + ind\ \mathfrak{g}).$$

The greatest practical interest is sets of polinomials.

- Mishchenko-Fomenko's conjecture: on the dual space $\mathfrak{g}^*$ of any Lie algebra $\mathfrak{g}$, there is a complete set of polynomials that are in involution.

- Generalized Mishchenko-Fomenko's conjecture (argument shift conjecture): on the dual space $\mathfrak{g}^*$ of any Lie algebra $\mathfrak{g}$ there is a complete set of polynomials in bi-involution, i.e, a set that is simultaneously in involution with respect to $\mathcal{A}_x$ and $\mathcal{A}_a$.

The first hypothesis was proved by Sadetov in 2004 [1], but the sets obtained by him did not always turn out to be in involution with respect to the bracket with a frozen argument. It is worth noting that when applying the general method for constructing sets in bi-involution (Mischenko–Fomenko's argument shift method), firstly, the resulting sets are not complete for all Lie algebras (as, for example, for semisimple Lie algebras), secondly, not for all values of the parameter a, set of functions are functionally independent.
This report will talk about the generalization of Sadetov's method. Sadetov's method based on the lemma:

Lemma(Sadetov, Bolsinov)[2]

Let $\mathfrak{g}$ be a Lie algebra over field $\mathbb{K}$ of characteristic 0, then algebra belongs to one of the types:

- $\mathfrak{g}$ *has a non-one-dimensional commutative ideal or a one-dimensional ideal that does not included in the center of* $\mathfrak{g}$ *;*

- *in* $\mathfrak{g}$ *there exists an ideal isomorphic to the Heisenberg algebra, and the center of* $\mathfrak{g}$ *coincides with the center of the ideal;*

- $\mathfrak{g} = L \bigoplus \mathbb{K}$, *$L$ is simesimple;*

- $\mathfrak{g}$ *is semisimple;*

Based on Sadetov's algorithm, we can proof the following theorems.

Theorem (for the first type)

Let $\mathfrak{g}$*, has a non-one-dimensional commutative ideal*$\mathfrak{h} \subset \mathfrak{g}$ *or a one-dimensional ideal* $\mathfrak{h} \subset \mathfrak{g}$ *that does not included in the center of* $\mathfrak{g}$*. Let* a *be some covector such that* $a|_{\mathfrak{h}} = 0$*. Then there exists a Lie algebra* $\mathfrak{L}_{\mathfrak{h}}$ *and a functional* $a^{'}$ *on it such that the problem of constructing a complete bi-involutive set in* $\mathfrak{g}$ *with respect to* a*, reduces to the same problem over* $\mathfrak{L}_{\mathfrak{h}}$ *and* $a^{'}$*. Moreover, the dimension of* $\mathfrak{L}_{\mathfrak{h}}$ *is less than the dimension of* $\mathfrak{g}$*.*

Theorem (for the second type)

Let in $\mathfrak{g}$ *exists an ideal* $\mathfrak{h}$ *isomorphic to the Heisenberg algebra , and the center of* $\mathfrak{g}$ *coincides with the center of the ideal,* a *is a covector such that* $\mathfrak{h} \subset St_{ad^*}(a)$*. Then there exists a Lie algebra* $\mathfrak{b} \subset \mathfrak{g}$ *such that the problem of constructing a complete bi-involutive set for* $\mathfrak{g}$ *with respect to* a *reduces to the same problem over* $\mathfrak{b}$ *with respect to* $a|_{\mathfrak{b}}$*.*

Despite the non-triviality of the methods by which the theorems were proved, they help to construct complete bi-involutive sets of polynomials for many cases for which other methods do not work.

The author is a stipendiat of the Theoretical Physics and Mathematics Advancement Foundation "BASIS".

REFERENCES

1. *Sadetov S. T.*, A proof of the Mishchenko–Fomenko conjecture, Dokl. Math., textbf70, No. 1, 635–638 (2004).
2. *Bolsinov A. V.*, Complete commutative subalgebras in polynomial Poisson algebras: a proof of the Mischenko–Fomenko conjecture, Sbornik: Mathematics, **200**, No. 12, 1731 (2009).

NONLOCAL DYNAMICS OF TWO COUPLED OSCILLATORS WITH DELAYED FEEDBACK

I. S. Luzin

Yaroslavl State University, Yaroslavl, Russia;
mr.luzin.ilya@yandex.ru

The talk will consider the following model of differential equations with delay

$$\begin{cases} \dot{u}_1 + u_1 = \lambda F(u_1(t-1)) + \gamma(u_2 - u_1), \\ \dot{u}_2 + u_2 = \lambda F(u_2(t-1)) + \gamma(u_1 - u_2), \end{cases} \tag{1}$$

where the functions u_1 and u_2 are real and scalar.

The function $F(x)$ is defined as follows:

$$F(x) = \begin{cases} b,\, x < -1, \\ f(x),\, x \in [-1, 1], \\ d,\, x > 1, \end{cases}$$

$$\lambda \gg 1, \quad \gamma \in (-\frac{1}{2}, 0) \cup (0, +\infty).$$

The function $f(x)$ is nonlinear, $f(x) \neq 0$ on any segment of non-zero length.

In our work for the model under consideration, we numerically calculate solutions and analyze the results obtained, as well as analytically construct solutions in some cases.

The following types of solutions to the model were found: tending to a positive constant λd, tending to a negative constant λb, tending to different constants, homogeneous cycle and inhomogeneous cycle.

We analytically solve the system (1) in the case of $b > 0$ and $d < 0$ for a specific function f. Homogeneous periodic and inhomogeneous periodic solutions were analytically found.

EMBEDDING OF INVARIANT MANIFOLDS OF POLAR FLOWS WITH TWO SADDLES

D. A. Maksimov

HSE University, Nizhny Novgorog, Russia;
damaksimov_2@edu.hse.ru

A smooth flow f^t on a closed smooth manifold M^n of dimension n is *polar* if it is structurally stable and its non-wandering set Ω_{f^t} consists of exactly one source, one sink and some finite number of saddles. We consider a polar flow on M^4 with exactly two saddle equilibria σ_1, σ_2 and assume that the dimension of unstable manifolds of these equilibria equal one and two, correspondingly.

Let us recall that a closed manifold $X \subset M^n$ of dimension m is called *locally flat* if for any point $x \in X$ there exist an open set $U_x \subset M^n, x \in U_x$ and homeomorphism $h\colon U_x \to \mathbb{R}^n$ such that $h(X \cap U_x)$ is a coordinate hyperplane $\mathbb{R}^m \subset \mathbb{R}^n$. If condition of locally flateness fail at a point $x \in X$ then the manifold X is called *wild*.

Theorem 1.

1. *M^4 is homeomorphic to sphere S^4;*
2. *$W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is not empty;*
3. *the closure of $W^u_{\sigma_1}, W^s_{\sigma_2}$ are a locally flat two-dimensional sphere and simple closed curve, correspondingly.*

For $n = 2$ the manifold M^n carrying the polar flow with two saddles is either the torus or Klein Bottle, for $n = 3$ is Lens space, so this is surprise that for $n = 4$ the carrying manifold is unique up to homeomorphism. Item 3 of the Theorem contrasts to Theorem 6 of [1], where an example of polar flow with two saddles with wild closures of separatricies was provided (both separatricies are two-dimensional).

Researh was supported by Program "Scientific Foundation of the National Research University Higher School of Economics" in 2023-2024 (project No. 23-00-028).

REFERENCES

1. *E. V. Zhuzhoma, V. S. Medvedev,* "Morse-Smale systems with few nonwandering points," Topology and its Applications, **160**:3, 498–507 (2013).

EVOLUTION OF MOTION OF A SOLID SUSPENDED ON A THREAD IN A HOMOGENEOUS GRAVITY FIELD

A. P. Markeev

Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Russia; anat-markeev@mail.ru

The plane motion of a solid in a uniform gravity field is studied. The solid is suspended on a weightless inextensible thread, which remains stretched during the motion. It is assumed that the length of the thread is large ($\sim \varepsilon^{-1/2}$) and the distance from the point of solid suspension to its center of gravity is small ($\sim \varepsilon$). The equations of motion are presented as equations of a system with one rapidly rotating phase. This system is analyzed using classical perturbation theory and KAM theory. It is shown that for all values of time, the movement differs little (by $\sim \varepsilon$) from the slow oscillations of the thread in the vicinity of the descending vertical and the solid rotation relative to the suspension point with an almost constant angular velocity. The measure

of the set of motions, different from the above motions, is estimated from above by the value of the order of $\exp(-c/\varepsilon)$ ($c > 0 - const$).

The author was supported by the state assignment (registration No. 123021700055-6) at the Ishlinskii Institute for Problems in Mechanics, RAS.

REFERENCES

1. *Arnold V. I.,Kozlov V. V.,Neishtadt A. I.,* Mathematical Aspects of Classical and Celestial Mechanics, Springer, Berlin.Heidelberg, (2006).
2. *Neishtadt A. I.,* “Estimates in the Kholmogorov theorem on conservation of conditionally periodic motions,” J. Appl. Math. Mech., **45**, No. 6, 766–772 (1981).
3. *Ishlinky A. Yu.,Storozhenko V. A.,Temchenko M. E.,* Rotation of a Rigid Body on a String and Adjacent Problems, Nauka, Moscow,(1991).
4. *Ishlinky A. Yu.,Storozhenko V. A.,Temchenko M. E.,* "Dynamics of rigid bodies rotating rapidly on a string and related topics (survey)," Int. Appl. Mech., **30**, No. 8, 557–581 (1994).
5. *Mirer S. A.,Sarychev V. A.,* "On stationary motions of a body on a string suspension,"in: Non-Linear Mechanics, Ed. by V. M. Matrosov V. M., Rumyantsev V. V., Karapetyan A. V., Fizmatlit, Moscow, 2001, pp. 281–322.
6. *Markeev A. P.,* "On periodic motions of a rigid body suspended on a thread in a uniform gravity field," Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, **29**, No. 2, 245–260 (2019).
7. *Ivanov A. P.,* "On the stability of permanent rotations of a body suspended on a string in the presence of shock interactions," Izv. Akad. Nauk SSSR Mekh. Tverd. Tela, No. 6, 47–50 (1985).
8. *Markeev A. P.,* "Stability of a periodic motion of a rod suspended by an ideal thread," Mech. Solids, **42**, 497–506 (2007).

OSCILLATORY MODES IN THE SIMPLEST RING OF AUTOGENERATORS WITH ASYMMETRIC NONLINEARITY

E. A. Marushkina

P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;
marushkina-ea@yandex.ru

Consider a system that simulates the interaction of three nonlinear autogenerators connected in a ring with unidirectional coupling:

$$\ddot{u}_j + \frac{d}{dt}\,(\varepsilon u_j + \alpha u_j^2 + u_j^3) + u_j + \mu\, g(u_{j-1}) = 0\,, \quad j = 1, 2, 3\,, \qquad (1)$$

where $u_0 = u_3$, $\mu > 0$, and the sign of the parameters ε and α is arbitrary. The coupling function $g(u)$ is given by the equality

$$g(u) = u \exp\left(-\frac{u^n}{nb^n}\right), \tag{2}$$

where $b > 0$ is fixed.

For sufficiently small values of ε and $\mu = \nu\varepsilon$ the local theory described in the article [1] (see also [2]) applies to the system (1). At $\mu = \varepsilon = 0$ the critical case of three pairs of purely imaginary roots is realized. To find the normal form of the system (1) replacement

$$u_j = \sqrt{\mu}u_{j1}(t,\tau) + \mu u_{j2}(t,\tau) + \mu^{3/2}u_{j3}(t,\tau) + \dots, \quad j = 1,2,3,$$

was used. Here $\tau = \mu t$, the functions $u_{jk}(t,\tau)$ are 2π–periodic by t, and

$$u_{j1} = z_j(\tau)\exp(it) + \bar{z}_j(\tau)\exp(-it), \quad j = 1,2,3.$$

The complex amplitudes of z_j are to be determined during the execution of the algorithm. From the conditions of solvability of problems for $u_{j3}(t,\tau)$ in the class 2π–periodic by t functions at the third step of the algorithm, the following normal form is obtained:

$$z_j' = -\nu\frac{z_j}{2} + \frac{i}{2}z_{j-1} + dz_j|z_j|^2, \quad j = 1,2,3, \tag{3}$$

where $d = -\frac{3}{2} + \frac{2\alpha^2}{3}i$. In the article [1] the conditions for the existence and stability of self-similar cycles of the system (3) are found. The specified result is obtained for the case $\alpha \neq 0$. At sufficiently small values of the parameters ε and μ it is shown that the system (1) with a coupling function of the form (2) has an orbitally asymptotically stable cycle branching from the equilibrium state.

Numerical analysis of the system (1), (2) in order to find its nonlocal oscillatory modes, was carried out at values $\varepsilon = 0.001$, $b = 0.9$. The parameter α was selected in three different types: $\alpha = 0$ and $\alpha = \pm 0.5$, μ was taken as a bifurcation parameter. As a result, the values of the asymmetry parameter when the system generates periodic and chaotic oscillations are found. Moreover, numerical analysis of nonlocal dynamics has shown a significant dependence: the presence of chaotic oscillatory modes on the degree of asymmetry of a nonlinear element (tunnel diode), and the important role of nonlinear coupling between partial systems. The main difference between the cases of an asymmetric characteristic of a nonlinear element and the case of $\alpha = 0$ is that there are no symmetric or self-symmetric attractors. The next difference is in the size of the periodic dynamics area: the intervals of existence and stability of cycles expand with an increase in the value of $|\alpha|$.

Remark. The author is grateful to S. D. Glyzin for problem statement.

This work was supported by the Russian Science Foundation (project No. 21-71-30011), https://rscf.ru/en/project/21-71-30011/.

REFERENCES

1. *Glyzin S. D., Kolesov A. Y., Rozov N. K.* "Chaos phenomena in a circle of three unidirectionally connected oscillators," Computational Mathematics and Mathematical Physics, **46**, No. 10, 1724–1736 (2006).
2. *Kolesov A. Y., Rozov N. K.* Invariantnye tory nelinejnyh volnovyh uravnenij, Fizmatlit, 408 pp. (2004). (In Russ.)

POPULATION DYNAMICS OF RECURRENT SPIKING NEURAL NETWORKS: MACHINE LEARNING AND COMPUTATIONAL NEUROSCIENCE

O. V. Maslennikov

A.V. Gaponov-Grekhov Institute of Applied Physics of the R.A.S., Nizhny Novgorod, Russia; olmaov@ipfran.ru

In this work, we propose recurrent spiking neural networks as functional models for performing multiple cognitive-like tasks. The target tasks are motivated by popular cognitive neuroscience experiments and are characterized by a stimulus-response structure. Our work extends the framework of functional neural networks, which has recently become popular in the computational neuroscience community as a promising tool for designing models with predefined constraints that, after training, are capable of completing cognitive-like target tasks. In this framework, the neural network is trained on the cognitive functions in a supervised manner, and after that, the network is reverse engineered to relate the obtained input-output mappings with features of neural activity, thus providing insight into the mechanisms of computation through dynamics.

We design spiking recurrent neural networks that are, on the one hand, more biologically relevant in terms of neuronal dynamics than rate-based ones and, on the other hand, lie in the actively developing framework of neuromorphic computing, which aims at developing next-generation energy effective devices. By using supervised learning techniques, the initially random spiking networks are trained to perform multiple cognitive-like tasks depending on the task coding input. The trained neural networks are treated as multidimensional dynamical systems, and the mechanisms that impact the multitasking performance are analyzed. We found several features that are qualitatively consistent with experimental findings observed when studying task-performing animals. First, task-specific functional clusters of neurons that fire preferentially during particular tasks or trial phases emerged. Second, neurons with high mixed selectivity appeared, and they generated activity depending on various factors in a complicated manner. Third, we confirmed the principles of computation through dynamics, where special trajectories that emerged in the population phase space were responsible for information processing.

The work is supported by the Russian Science Foundation, project 19-12-00338.

REFERENCES

1. *Pugavko M. M., Maslennikov O. V., Nekorkin V. I.* "Multitask computation through dynamics in recurrent spiking neural networks," Scientific Reports, **13**, 3997 (2023).
2. *Maslennikov O. V., Pugavko M. M., Shchapin D. S., Nekorkin V. I.* "Nonlinear dynamics and machine learning of recurrent spiking neural networks," Physics-Uspekhi, **65**, No. 10, 1020-1038 (2022).

SIMULATION OF BUSINESS AND FINANCIAL CYCLES

V. V. Matrosov

National Research Lobachevsky State University of Nizhny Novgorod
23, Gagarin ave., Nizhny Novgorod 603022, Russia;
matrosov@rf.unn.ru

The work is devoted to the study of the dynamical processes and in particular the phenomenon of the synchronization in an ensemble of coupled chaotic economic oscillators. The nonlinear model of economic oscillator as the system of automatic control are considered. Such kind of general economic models are unsuitable for getting some concrete economic estimations and recommendations. But such kind models are very useful for a development the theory of the economic cycles, theory of the generation, interactions, synchronization of the cycles and so on. Our numerical experiments demonstrated a good enough qualitative similarity of an chaotic economic oscillations in our model and real economic cycles. The phenomen of the synchronization of the chaotic oscillations in the ensemble of coupled economic oscillators are considered, however the accuracy of the synchronization depends with couplings essentially [1-3].

The work is supported by the Ministry of Science and Higher Education of the Russian Federation (project No. FSWR-2023-0031).

REFERENCES

1. *Matrosov V.V., Shalfeev V.D.* "Simulation of business and financial cycles: Self-oscillation and synchronization," Izvestiya VUZ. Applied Nonlinear Dynamics, **29**, No. 4, 515–537 (2021).
2. *Matrosov V.V., Shalfeev V.D.* "Simulation of the business-cycle synchronization processes in an ensemble of coupled economic oscillators," Radiophysics and Quantum Electronics., **64**, No. 10, 750–759 (2022).
3. *Matrosov V.V., Shalfeev V.D.* "Coupled economic oscillations - synchronization dynamical model," Izvestiya VUZ. Applied Nonlinear Dynamics, **31**, No. 3, 254–270 (2023).

APPLIED COMBINATORICS OF UNIVERSAL ENVELOPING LIE ALGEBRAS

D. V. Millionshchikov

Lomonosov Moscow State University, Moscow, Russia
Gubkin Oil and Gas University, Moscow, Russia;
mitia_m@hotmail.com

The formalism of universal enveloping Lie algebras turns out to be effective and useful in a number of problems of mathematical physics related to: a) symmetries of nonlinear hyperbolic systems of partial differential equations [1]; b) Van Vleck operator canonical perturbation theory in high-resolution molecular laser spectroscopy [2,3].

The talk will focus on these two cycles of problems. We will discuss the appearance of some well-known and new combinatorial polynomials in problems related to the normal ordering of differential operators in quantum mecanics.

This work is supported by the grant of Moscow Center of Fundamental and Applied Mathematics.

REFERENCES

1. *Millionshchikov D.V., Smirnov S.V.* "Characteristic algebras and integrable systems of exponential type", Ufa Math. Journal, **13**:2, 44–73 (2021).
2. *Xuanhao C., Krasnoshchekov S., Pupyshev V., Millionshchikov D.* "Normal ordering of the su(1,1) ladder operators for the quasi-number states of the Morse oscillator", Physics Letters, Section A, **384**, 1–8 (2020).
3. *Xuanhao C., Efremov I., Millionshchikov D., Krasnoshchekov S.* "Normal Ordering of the Angular Momentum Cylindrical Ladder Operators and their Products with Wigner D-functions", Journal of Chemical Physics, https://doi.org/10.1063/5.0142809, (2023)

DYNAMICS OF DELAY DIFFERENTIAL EQUATION WITH SIMPLE BEHAVIOR AT INFINITY

E. V. Morozova

P. G. Demidov Yaroslavl State University, Russia;
evgenia24morozova@gmail.com

Consider delay differential equation

$$\dot{x} + x = \alpha \cdot F\left(x(t-T)\right), \tag{1}$$

where

$$F(x) = \begin{cases} 1, & x \geqslant 1 \\ x, & x \in (-1;1) \\ -1, & x \leqslant -1. \end{cases}$$

Out goal is to study the behaviour of solutions of the equation (1) for various values of α.

Consider the initial conditions $\varphi(t) \in C_{[-T,0]}$ such that

$$\begin{cases} \varphi(t) \geqslant 1, \\ \varphi(0) = 1. \end{cases} \tag{2}$$

The solution of (1) on a segment $[0;T]$ is

$$x(t) = (1-\alpha)e^{-t} + \alpha. \tag{3}$$

Further form of the solution depends on the parameter α.
The following results take place.

Lemma 1. *In case $\alpha < \alpha_* < -1$, where $\alpha_*^2 = 1 + \omega^2$, and ω is minimal positive root of the equation $\omega = -\tan \omega T$, solutions with initial conditions (2) tend to periodic.*

Lemma 2. *Let $\alpha > 1$, then on a half-interval $[T; +\infty)$ solutions with initial conditions (2) are (3) and tend to α.*

Note that if initial conditions are

$$\begin{cases} \varphi(t) \leqslant -1, \\ \varphi(0) = -1, \end{cases}$$

solutions tend to $-\alpha$.

Lemma 3. *Let $\alpha \in (\alpha_0; 1)$, where $\alpha_0 = \dfrac{e^{-T}+1}{e^{-T}-1} < -1$, then solutions tend to zero.*

LATTICE EQUATIONS AND SEMICLASSICAL ASYMPTOTICS

V. E. Nazaikinskii[1], **V. L. Chernyshev**[2], **A. V. Tsvetkova**[3]

[1]*Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia;* nazaikinskii@yandex.ru

[2]*HSE University, Moscow, Russia;* vchern@gmail.com

[3]*Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia;* moskal_1@mail.ru

We present the results of our recent paper [1]. This paper deals with an important class of linear equations with shifts in the argument, namely, equations on a uniform rectangular lattice with small step h in $\mathbb{R}^n$. In the case of functions of continuous argument, equations with shifts can be written as h-pseudodifferential equations [2] with symbols 2π-periodic in the momenta. This representation can also be given meaning for functions of a discrete argument, although differentiation operators are not defined for lattice functions. The phase space of such equations is the product $\mathbb{R}^n \times T^n$. Developing Maslov's ideas [2, 3], we construct a canonical operator on Lagrangian submanifolds of this phase space with values in the space of lattice functions. Compared with the classical version [4] of the canonical operator and the new formulas introduced in [5, 6], the construction involves a number of new features.

As an example, we give equations on a two-dimensional lattice that arise in quantum theory (the Feynman checkers model [7, 8]) and in the problem on the propagation of wave packets on a homogeneous tree [9].

The research of V. L. Chernyshev was carried out within the framework of the HSE University Basic Research Program. The research of V. E. Nazaikinskii was supported by the Russian Foundation for Basic Research under grant no. 21-51-12006. The work of A. V. Tsvetkova was supported by the Ministry of Science and Higher Education of the Russian Federation within the framework of the Russian State Assignment under contract no. 123021700044-0.

REFERENCES

1. *Nazaikinskii V. E., Chernyshev V. L., Tsvetkova A. V.* "Lattice equations and semiclassical asymptotics," Russ. J. Math. Phys., **30**, No. 2, 152–164 (2023).
2. *Maslov V. P.,* Operational Methods, Mir, Moscow (1976).
3. *Maslov V. P.,* "The characteristics of pseudo-differential operators and difference schemes," in: Actes du Congrès International des Mathématiciens, ICM, Nice, 1971, pp. 755–769.
4. *Maslov V. P., Fedoriuk M. V.,* Semi-Classical Approximation in Quantum Mechanics, D. Reidel, Dordrecht (1981).
5. *Dobrokhotov S. Yu., Nazaikinskii V. E., Shafarevich A. I.,* "New integral representations of the Maslov canonical operator in singular charts," Izv. Math., **81**, No. 2, 286–328 (2017).
6. *Dobrokhotov S. Yu., Nazaikinskii V. E., Shafarevich A. I.,* "Efficient asymptotics of solutions to the Cauchy problem with localized initial data for linear systems of differential and pseudodifferential equations," Russ. Math. Surveys, **76**, No. 5, 745–819 (2021).
7. *Feynman R. P., Hibbs A. R.,* Quantum Mechanics and Path Integrals, McGraw-Hill, New York (1965).
8. *Skopenkov M. B., Ustinov A. V.,* "Feynman checkers: towards algorithmic quantum theory," Russ. Math. Surveys, **77**, No. 3, 445–530 (2022).
9. *Tsvetkova A. V., Shafarevich A. I.,* "The Cauchy problem for the wave equation on homogeneous trees," Math. Notes, **100**, No. 6, 862–869 (2016).

EXISTENCE AND STABILITY OF PERIODIC CONTRAST STRUCTURES IN TIKHONOV-TYPE SYSTEMS

N. N. Nefedov

Lomonosov Moscow State University, Moscow, Russia;
nefedov@physics.msu.ru

We consider a singularly perturbed problem of the form:

$$
\begin{aligned}
&\varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}\Big)-g(u,v,x,t,\varepsilon)=0,\\
&\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial t}-f(u,v,x,t,\varepsilon)=0,\ x\in(0,1),\ t\in R,\\
&\frac{\partial u}{\partial x}(0,t,\varepsilon)=u^0(t),\ \frac{\partial u}{\partial x}(1,t,\varepsilon)=u^1(t),\ t\in R,\\
&v(0,t,\varepsilon)=v^0(t),\ v(1,t,\varepsilon)=v^1(t),\ t\in R,\\
&u(x,t,\varepsilon)=u(x,t+T),\ v(x,t,\varepsilon)=v(x,t+T,\varepsilon),
\end{aligned}
$$

where $\varepsilon>0$ is a small parameter. Such systems naturally arise when modeling fast bimolecular reactions in the case when one of the sources (reaction, nonlinear source, interaction) is intense (on the order of $1/\varepsilon^2$), and the second is of order unity.

Conditions are obtained under which the problem has solutions with boundary and inner layers. The asymptotics of such solutions is constructed and their asymptotic stability in the sense of Lyapunov is proved.

The paper presents a further development of the results presented in the review [1].

The work was supported by the Russian Science Foundation (project N 23-11-00069).

REFERENCES

1. *Nefedov N. N.*, “Development of methods of asymptotic analysis of transitionlayers in reaction-diffusion-advection equations:theory and applications,” Computational Mathematics and Mathematical Physics, **61**, No. 12, 2068–2087 (2021).

ON SOME PROBLEMS OF DYNAMICAL NETWORKS

V. I. Nekorkin

A. V. Gaponov-Grekhov Institute of Applied Physics of the Russian Academy of Sciences, 46 Ul'yanov street, Nizhny Novgorod, 603950, Russia; vnekorkin@ipfran.ru

I will present a brief review of results of the study of the dynamics of oscillatory systems: adaptive Kuramoto networks, power grids, machine learning networks, systems with time delay, neural mass models, etc.

The work is supported by the Russian Science Foundation (grant no. 23-42-00038).

ON THE CRITICAL STATE FOR OSCILLATION OF FIRST-ORDER DELAY DIFFERENTIAL EQUATIONS

P. N. Nesterov[1], J. I. Stavroulakis[2]

[1]*Yaroslavl State University, Yaroslavl, Russia;*
p.nesterov@uniyar.ac.ru
[2]*Ariel University, Ariel, Israel;*
john.ioannis.stavroulakis@gmail.com

In this talk we discuss both new and classic results concerning the critical state of oscillation of the nonautonomous delay differential equation

$$x'(t) = -a(t)x(\tau(t)), \qquad t \geq t_0, \tag{1}$$

where $a(t)\colon \mathbb{R} \to \mathbb{R}$, $\tau(t)\colon \mathbb{R} \to \mathbb{R}$ are measurable, $a(t)$ is locally integrable and $\tau(t) \leq t$, $t \in \mathbb{R}$, $\lim_{t\to\infty} \tau(t) = \infty$. By a solution of (1) we mean a function

$$x(t)\colon \big[\inf\{\tau(t)\colon t \geq t_0\}, +\infty\big) \to \mathbb{R},$$

which is absolutely continuous and satisfies (1) on $[t_0, \infty)$. Such a solution $x(t)$ of Eq. (1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

Proposition 1. *If $\tau(t) = t - 1$ and $a(t) \leq c \in \left[0, \frac{1}{e}\right]$ then (1) is nonoscillatory. If $\tau(t) = t - 1$ and $a(t) \geq c \in \left(\frac{1}{e}, +\infty\right)$ then (1) is oscillatory.*

The above Proposition was proven by Myshkis [10, 11] for continuous parameters. It was subsequently generalized in several papers, e.g., [7, 9]. It should be noted that

$$\limsup_{t\to\infty} \int\limits_{\tau(t)}^{t} a(s)ds < \frac{1}{e}$$

implies nonoscillation and if τ is monotone

$$\limsup_{t\to\infty} \int\limits_{\tau(t)}^{t} a(s)ds > \frac{1}{e}, \quad \liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} a(s)ds \geq \frac{1}{e}$$

implies oscillation [13]. The case of

$$\limsup_{t\to\infty} \int\limits_{\tau(t)}^{t} a(s)ds = \liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} a(s)ds = \frac{1}{e}$$

is referred to as the *critical* state, as either oscillation or nonoscillation is possible. Some of the earliest results on the critical case were obtained in [1, 3, 4, 8]. Already in [3], one noted that several oscillation criteria for the critical state are similar to the Kneser–Hille oscillation criteria [5, 6] for the second order ODE

$$y''(t) + f(t)y(t) = 0.$$

This was rigorously demonstrated in [14, 15, 16], proving the following.

Proposition 2. *Assume $a(t) \geq \dfrac{1}{e}$, $t \in \mathbb{R}$. The second order ODE*

$$y''(t) + 2e(a(t) - 1/e)y(t) = 0$$

oscillates if and only if the same holds for the delay equation

$$x'(t) = -a(t)x(t-1).$$

In parallel to the above, the following result was obtained in [3] (building upon [2]) for the case where $\tau(t) = t-1$ and $a(t)$ is continuous and oscillates around $1/e$.

Proposition 3. *Assume $\tau(t) = t - 1$ and that there exists a continuous function $a_0(t)\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le a_0(t) \le a(t)$, $t \in \mathbb{R}$,*

$$\int_t^{t+1} a_0(s)ds \quad \text{is bounded and} \quad \int^{\infty} \frac{a_0(s)}{s} ds = \infty.$$

Assume that $\liminf_{t\to\infty} \Phi(t) = 1$ *and*

$$D := \liminf_{t\to\infty} \big\{[\Phi(t) - 1]t^2\big\} > 0,$$

where

$$\Phi(t) := t\psi(t) \exp \int_t^{t+1} \frac{a_0(s)}{s\psi(s)} ds, \quad \psi(t) := \int_t^{t+1} \frac{a_0(s)}{s} ds.$$

Then all solutions of Eq. (1) *oscillate.*

More recently the case when $a(t)$ and $\tau(t)$ may oscillate around the critical values was studied in [12]. It is assumed that the functions $a(t)$ and $\tau(t)$ have the following asymptotic expansions as $t \to \infty$:

$$a(t) = \frac{1}{e} + a_1(t)t^{-\rho} + a_2(t)t^{-2\rho} + \cdots + a_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}),$$
$$t - \tau(t) = 1 + q_1(t)t^{-\rho} + q_2(t)t^{-2\rho} + \cdots + q_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}),$$

where $\rho > 0$ and $k \in \mathbb{N}$ is chosen such that $(k+1)\rho > 1$. Functions $a_j(t)$, $q_j(t)$, $j = 1, \ldots, k+1$, are finite trigonometric polynomials. The author constructs the asymptotics for solutions of Eq. (1) using the perturbation methods technique, averaging method and the ideas of center manifold theory. The leading part of the constructed asymptotic representations is then used to solve the oscillation problem for the studied equation.

In the present talk, we discuss possible extensions of comparison results such as Proposition 2 to the case where $a(t)$ and $\tau(t)$ oscillate around the critical value, using the methods of [12].

The first author was supported by the Russian Science Foundation (project No. 21-71-30011), https://rscf.ru/en/project/21-71-30011/. The second author acknowledges the support of Ariel University.

REFERENCES

1. *Diblík J.,* “Positive and oscillating solutions of differential equations with delay in critical case”, J. Comput. Appl. Math., **88**, No. 1, 185–202 (1998).
2. *Domshlak Y.,* Metod sravneniya po Shturmu v issledovanii povedeniya reshenij differentsial’no-operatornykh uravnenij (in Russian) [Sturmian Comparison Method in Investigation of the Behavior of Solutions for Differential-Operator Equations], ELM, Baku, USSR (1986).
3. *Domshlak Y., Stavroulakis I. P.,* “Oscillations of first-order delay differential equations in a critical state”, Appl. Anal., **61**, No. 3-4, 359–371 (1996).
4. *Elbert Á., Stavroulakis I. P.,* “Oscillation and nonoscillation criteria for delay differential equations”, Proc. Amer. Math. Soc., **123**, No 5, 1503–1510 (1995).
5. *Hille E.,* “Non-oscillation theorems”, Trans. Amer. Math. Soc., **64**, No. 2, 234–252 (1948).
6. *Kneser A.,* “Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen”, Math. Ann., **42**, 409–435 (1893).
7. *Koplatadze R. G., Chanturiya T. A.,* “O koleblyushchihsya i monotonnyh resheniyah differencial’nyh uravnenij pervogo poryadka s otklonyayushchimsya argumentom (in Russian) [Oscillating and monotone solutions of first-order differential equations with deviating argument]”, Differ. Uravn., **18**, No. 8, 1463–1465 (1982).
8. *Kozakiewicz E.,* “Conditions for the absence of positive solutions of a first order differential inequality with a single delay”, Arch. Math. (Brno), **31**, No. 4, 291–297 (1995).
9. *Kwong M. K.,* “Oscillation of first-order delay equations”, J. Math. Anal. Appl., **156**, No. 1, 274–286 (1991).
10. *Myshkis A. D.,* “O resheniyah linejnyh odnorodnyh differencial’nyh uravnenij pervogo poryadka ustojchivogo tipa s zapazdyvayushchim argumentom (in Russian) [On solutions of linear homogeneous differential equations of the first order of stable type with a retarded argument]”, Mat. Sb. (N.S.), **28(70)**, No. 3, 641–658 (1951).
11. *Myshkis A. D.,* Linejnye differencial’nye uravneniya s zapazdyvayushchim argumentom (in Russian) [Linear differential equations with retarded argument], Nauka, Moscow (1972).
12. *Nesterov P.,* “On oscillation of solutions of scalar delay differential equation in critical case”, Electron. J. Qual. Theory Differ. Equ., No. 59, 1–31 (2022).
13. *Pituk M., Stavroulakis I. P., Stavroulakis J. I.,* “Explicit values of the oscillation bounds for linear delay differential equations with monotone argument”, Commun. Contemp. Math., **25**, No. 3, paper No. 2150087 (2023).
14. *Shen J., Tang X. H.,* “New nonoscillation criteria for delay differential equations”, J. Math. Anal. Appl., **290**, No. 1, 1–9 (2004).
15. *Tang X. H., Yu J. S.,* “Oscillation of first order delay differential equations”, J. Math. Anal. Appl., **248**, No. 1, 247–259 (2000).
16. *Tang X. H., Yu J. S., Wang Z. C.,* “Comparison theorems of oscillation of first order delay differential equations in a critical state with applications”, Kexue Tongbao, **44**, No. 1, 26–30 (1999).

ASYMPTOTICS OF THE SOLUTION TO THE CAUCHY PROBLEM FOR A SINGULARLY PERTURBED DIFFERENTIAL OPERATOR TRANSFER EQUATION WITH LOW DIFFUSION AND MANY SPATIAL VARIABLES

A. V. Nesterov[1], **A. V. Zaborsky**

[1]*Plekhanov Russian University of Economics, Moscow, Russia;*
andrenesterov@yandex.ru

The asymptotics of the solution of the Cauchy problem for a singularly perturbed differential operator equation is constructed

$$\varepsilon^2(U_t + \sum_{i=1}^{m} D_i(p)U_{x_i}) = L_pU + \varepsilon F(U,p) + \varepsilon^3 \sum_{i=1}^{m} B_i(p)U_{x_ix_i}, \qquad (1)$$

$$U(\bar{x},0,p) = \omega\left(\bar{x}\varepsilon^{-1},p\right) \qquad (2)$$

$0 < \varepsilon << 1$. The linear operator L_p acting on variable p has a single zero eigenvalue $\lambda = 0$,$\forall\lambda \neq 0$ $Re\lambda < 0$. The initial conditions $|\omega^{(k)}(\bar{z},p)| \leqslant Ce^{-\sigma\|\bar{z}\|^2}, \sigma > 0, \forall k = 0,1,....$ This work is a continuation of the works [1-2].

The asymptotics of the solution of problem (1)-(2) is constructed in the form

$$U(\bar{x},t,p,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i(s_i(\bar{\zeta},t,p) + p_i(\bar{\xi},\tau,p)) + R = U_N + R \qquad (3)$$

where the variables $\bar{\xi}$, τ, $\bar{\zeta}$ are expressed in terms of the task data. Under certain conditions on the functions $D_i(p), B_i(p)$, the main term of the asymptotics of the solution is described by a BKdV type equation generalised to a multidimensional space.

REFERENCES

1. *A. V. Nesterov*, “The effect of weak mutual diffusion on transport processes in a multiphase medium,” Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki, **61**, No. 3, 51–528 (2021).
2. *A. V. Nesterov and A. V. Zaborsky* , “On the asymptotic behavior of the solution to the Cauchy problem for singularly perturbed differential operator equation transfer with low diffusion,” Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki, **63**, No. 2, 273–281. (2023).
3. *A. B. Vasilyeva and B. F. Butuzov* , “Singularly perturbed equations in critical cases,” Moscow State University Publishing, 262 pg. (1978).

ON THE SOLUTION OF THE 33 PALIS-PUGH PROBLEM FOR GRADIENT-LIKE ORIENTATION-CHANGING MORSE-SMALE DIFFEOMORPHISMS

E. V. Nozdrinova

HSE University, Nizhny Novgorod, Russian Federation;
maati@mail.ru

A diffeomorphism of a closed surface $f : M^2 \to M^2$ is called *gradient-like* if its nonwandering set Ω_f consists of a finite number of hyperbolic periodic points and the invariant manifolds of distinct saddle points do not intersect. One of the problems formulated in 1975 by J. Palis and C. Pugh [1] is the description of a criterion for determining whether two such isotopic diffeomorphisms are connected by a stable arc (which retains its qualitative properties at small perturbations) in the space of diffeomorphisms.

Some obstructions to the existence of stable arcs between gradient-like diffeomorphisms of a surface were first discovered by P. Blanchard [2] (also see the survey [3] on currently known obstructions to the existence of stable arcs between diffeomorphisms of manifolds) and were expressed in terms of periodic data of diffeomorphisms. For orientation-preserving gradient-like diffeomorphisms of the 2-sphere, a complete classification up to stable isotopy was obtained by E. Nozdrinova and O. Pochinka in [4]. In this paper, we present a similar classification for orientation-changing gradient-like diffeomorphisms of the two-dimensional sphere. In more detail.

Consider the circle S^1 as the equator of the 2-sphere S^2. Then a structurally stable circle diffeomorphism with exactly two periodic orbits of period $m \in \mathbb{N}$ and rotation number $\frac{k}{m}$ can be extended to a diffeomorphism $\psi_{k,m} : S^2 \to S^2$, which has a periodic source of period 2 at the north and south poles. Denote by $C_{k,m}$ the component of the stable isotopy connection of the diffeomorphism $\psi_{k,m}$ and by $C^-_{k,m}$ the component of the stable isotopy connection of the diffeomorphism $\psi^{-1}_{k,m}$. Denote by C_0 the stable isotopy connected component of the source-sink diffeomorphism $\psi_0 \in G$ with a nonwandering set consisting of exactly one source and one sink.

The main result is the following theorem.

Theorem 1.

Any orientation-changing gradient-like diffeomorphism of the two-dimensional sphere S^2 *belongs to one of the components* C_0, $C_{k,m}$, $C^-_{k,m}$, k, m *in* $\mathbb{N}$, $k < m/2$, $(k, m) = 1$.

REFERENCES

1. *Palis J., Pugh C. Fifty problems in dynamical systems, Lecture Notes in Math. 1975. Vol. 468. P. 345–353.*
2. *P. R. Blanchard, Invariants of the NPT isotopy classes of Morse-Smale diffeomorphisms of surfaces, Duke Mathematical Journal, 47:1 (1980), 33–46.*
3. *Medvedev T. V., Nozdrinova E., Pochinka O. Components of Stable Isotopy Connectedness of Morse – Smale Diffeomorphisms, Regular and Chaotic Dynamics. 2022. Vol. 27. No. 1. P. 77–97.*
4. *Nozdrinova E., Pochinka O. Solution of the 33rd Palis-Pugh problem for gradient-like diffeomorphisms of a two-dimensional sphere // Discrete and Continuous Dynamical Systems. 2021. Vol. 41. No. 3. P. 1101–1131.*

EXISTENCE AND STABILITY OF SOLUTIONS WITH AN INTERNAL TRANSITION LAYER OF THE REACTION-DIFFUSION-ADVECTION EQUATION WITH KPZ – NONLINEARITY

A. O. Orlov

Lomonosov Moscow State University, Moscow, Russia;
orlov.andrey@physics.msu.ru

We consider the following one-dimentional singularly perturbed quasilinear reaction-diffusion-advection problem:

$$\varepsilon^2 \frac{d^2u}{dx^2} = \varepsilon^2 A(u,x)\left(\frac{du}{dx}\right)^2 + f(u,x,\varepsilon), \quad x \in (-1;1), \quad u'(\pm 1, \varepsilon) = u^{(\pm)}, \tag{1}$$

where ε is a small parameter.

ODE equations with nonlinearities involving the scalar square of the unknown function gradient (known as Kardar–Parisi–Zhang (KPZ) nonlinearities) arise in various applications: population dynamics, free surface growth in polymer theory, nonlinear theory of thermal conductivity. Such equations are also of interest from a purely theoretical point of view, since they contain the squared first derivative of the unknown function. It's well known that this is the highest power exponent at which Bernstein-type conditions are fulfilled (the nonlinearity belongs to the class of Nagumo).

For this problem, conditions are obtained under which solutions with an inner transition layer are Lyapunov stable. We consider so called cirtical and non-critical cases. Existence theorems are proved using the asymptotic method of differential inequalities. Asymptotic approximations are constructed using Vasileva's method. Also we obtain the instability conditions.The proofs of solutions instability of are based on the construction of an unordered pair of upper and lower solutions and on the application of the Krein–Rutman theorem corollary. As a consequence of the theorem, we obtained an estimate for the main eigenvalue of the problem under consideration.

REFERENCES

1. *Nefedov N. N., Orlov A. O.,* "On unstable contrast structures in one-dimensional reaction-diffusion-advection problems with discontinuous sources," TMP, **215**, No. 1, 716–728 (2023).
2. *Vasil'eva A. B., Davydova M. A* "On a contrast steplike structure for a class of second-order nonlinear singularly perturbed equations," Computational Mathematics and Mathematical Physics, **38**, No. 6, 900–910 (1998).

ELLIPTIC ORTHOGONAL CURVILINEAR COORDINATE NETS AND SEPARATION OF VARIABLES IN THE LAPLACE OPERATOR

M. V. Pavlov

Lebedev Physical Institute, Moscow, Russia; mpavlov@itp.ac.ru

Separation of Variables for the Laplace equation is a well-known theory associated with elliptic orthogonal curvilinear coordinate nets and their various degenerations. The transformation from Cartesian coordinates to elliptic contains an important ingredient, i.e. the linear ordinary differential equation of second order. The aim of my talk is an integrability of this equation. We discuss a one-parametric solution and a corresponding integration method.

PARAMETERS ESTIMATION IN THE TRAFFIC FLOW MODEL

M. A. Pogrebnyak

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
pogrebnyakmaksim@mail.ru

This paper presents a parameters evaluation of the traffic flow mathematical model, which describes the movement of $N \in \mathbb{N}$ vehicles. The current model is an improvement of the one proposed in [1]. It is represented by a system of differential equations with a delay:

$$\begin{cases} \ddot{x}_1(t) = R_1\left[a_1\left(v_{max,1} - \dot{x}_1(t)\right)\right] - (1 - R_1)\left[H_1\dot{x}_1(t)\right], \\ \ddot{x}_n(t) = R_n\left[a_n\left(P_n - \dot{x}_n(t)\right)\right] - (1 - R_n)\left[H_n\dot{x}_n(t)\right], \\ x_n(t) = \lambda_n, \quad \dot{x}_n(t) = v_n, \quad \text{when } t \in [-\tau, 0], \end{cases} \tag{1}$$

where R_n — relay function as follows:

$$R_n = \begin{cases} 1, & \text{if } \Delta x_n(t,\tau) > (\tau + \tau_b)\dot{x}_n(t) + \dot{x}_n^2(t)/2\mu g + l_n, \\ 0, & \text{if } \Delta x_n(t,\tau) \le (\tau + \tau_b)\dot{x}_n(t) + \dot{x}_n^2(t)/2\mu g + l_n, \end{cases}$$

which describes the "acceleration-deceleration" switch; $\Delta x_n(t,\tau) = x_{n-1}(t-\tau) - x_n(t)$ — distance between adjacent vehicles ($x_0(t-\tau) = L$, where L — any distance); τ — driver response time; τ_b — brake system response time; μ — coefficient of friction; g — acceleration of gravity; l_n — sum of the safe distance between two adjacent vehicles and the length of the forward vehicle $l_n = l_{safe} + l_{veh,n-1}$ ($l_{veh,0} = 0$); $a_n > 0$ — the sensitivity coefficient characterizing the inverse time of matching speeds of two adjacent vehicles; $v_{max,n} > 0$ — maximum desired speed; P_n — logistic function as follows:

$$P_n = \frac{v_{max,n} - V_n}{1 + \exp[k_n(-\Delta x_n + S_n)]} + V_n,$$

describing the speed adjustment of the pursuing vehicle relative to the forward; V_n — function as follows:

$$V_n = \min\left(\dot{x}_{n-1}(t-\tau), v_{max,n}\right) \quad \text{when } n > 1, \quad V_1 = 0,$$

which does not allow to accelerate faster than maximum desired speed $v_{max,n}$; $k_n > 0$ — the logistic growth rate, showing the smoothness of the speed adjustment; S_n — parameter of the logistic function as follows:

$$S_n = (\tau + t_b)\dot{x}_n(t) + \dot{x}_n^2(t)/2\mu g + l_n + s_n,$$

where s_n describes the ceasing distance of the influence of the vehicle in front and it is equal to the distance traveled in τ time at the rate of convergence $s_n = \tau(\dot{x}_{n-1}(t-\tau) - \dot{x}_n(t))$; H_n — Heaviside function as follows:

$$H_n = \begin{cases} q_n \left(\dfrac{\dot{x}_{n-1}(t-\tau) - \dot{x}_n(t)}{\Delta x_n - l_n}\right)^2, & \text{if } q_n \left(\dfrac{\dot{x}_{n-1}(t-\tau) - \dot{x}_n(t)}{\Delta x_n - l_n}\right)^2 \le \mu g, \\ \mu g, & \text{if } q_n \left(\dfrac{\dot{x}_{n-1}(t-\tau) - \dot{x}_n(t)}{\Delta x_n - l_n}\right)^2 > \mu g, \end{cases}$$

which describes the vehicle braking, where $q_n > 0$ — intensity of braking ($\dot{x}_0(t-\tau) = 0$); λ_n — initial position of the vehicles; v_n — initial speed of the vehicles.

For example, let's describe the definition of a range of values a_n, which characterizes the time of matching the reverse speed. It is enough to take into account the only one vehicle acceleration, since a_n does not depend on the number of vehicles and the location in the flow. Consider the acceleration phase (R=1) of one vehicle. In this case, the model (1) will take the form as follows:

$$\begin{cases} \ddot{x}(t) = a\left(v_{max} - \dot{x}(t)\right) \\ x(0) = 0, \quad \dot{x}(0) = 0. \end{cases}$$

To simulate vehicle acceleration, a special computer program was developed that solves the equation using the Runge-Kutta method. The program solves the equation at various values of the coefficient a and measures the vehicle acceleration time to the v_{max} speed. On average, the acceleration time of a passenger car to $v_{max} = 100$ km/h speed is achieved in the range from 5 to 15 seconds [2]. Meanwhile, the values of the a coefficient are in the following range $a \in [0,31,0,92]$. Let's assume that the average time is 10 seconds, then in this situation the coefficient value will be 0.46.

The other parameters are defined similarly.

This work was supported by the Russian Science Foundation (project No. 21-71-30011), https://rscf.ru/en/project/21-71-30011/

REFERENCES

1. *Pogrebnyak M. A.,* Traffic Flow Model, Mathematical Models and Computer Simulations, Moscow (2023).
2. *Pozhidaev S. P.,* Estimation of the acceleration time of cars, Proceedings of the Orenburg State Agrarian University, Orenburg (2014).

ON THE DYNAMICS AND INTEGRABILITY OF THE ZIEGLER PENDULUM

I. Yu. Polekhin

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia;
Moscow Institute of Physics and Technology, Dolgoprudny, Russia;
Lomonosov Moscow State University, Moscow, Russia;
P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
ivanpolekhin@mi-ras.ru

We prove that the Ziegler pendulum — a double pendulum with a follower force — can be integrable, provided that the stiffness of the elastic spring located at the pivot point of the pendulum is zero and there is no friction in the system. We show that the integrability of the system follows from the existence of two-parameter families of periodic solutions. We explain the mechanism for the transition from the integrable to the chaotic dynamics and show that this mechanism is different from the typical phenomena observed in conservative Hamiltonian systems. The case in which the stiffnesses of the both springs are non-zero is briefly studied numerically. We show that the regular dynamics coexists with the chaotic dynamics.

MODELLING OF SPATIO-TEMPORAL STRUCTURES AT THE FLAME FRONT IN THE REDUCED MODEL OF HYDROGEN COMBUSTION

A. A. Polezhaev[1], **E. O. Yakupov**[2], **V. V. Gubernov**[3]

Lebedev Physical Institute, Russian Academy of Sciences, Moscow, Russia; [1]apol@lpi.ru, [2]edward.thrst@gmail.com, [3]v.v.gubernov@gmail.com

In chemical systems, in particular, in combustion ones, different space-time regimes are observed [1,2]. Thus, depending on the Lewis number, which determines the ratio of the efficiency of heat transfer to the diffusion of chemical components, either autowave or cellular structures can appear at the combustion front. In the case when the Lewis number is less than unity, a cellular structure or, in terms of nonlinear dynamics – a dissipative (Turing) structure, arises which manifests itself in the formation of an inhomogeneous temperature distribution in the plane of the flame front. For a Lewis number greater than unity, propagation instabilities and pulsations arise at the flame front, which lead to the formation of various unsteady combustion regimes.

The aim of the current study is to undertake a systematic reduction of the hydrogen combustion mechanism and then qualitatively separate a chemical mechanism causing the formation of nontrivial wave patterns. On the basis of a simple mathematical model obtained by reducing the complete model describing the kinetics of hydrogen combustion, the conditions for the formation of different types of structures were studied. The reduction of the detailed model was carried out taking into account the reaction rates known from the experiment. As a result, a model was obtained containing three equations for the most significant components — radicals H, HO_2 and oxygen O_2, as well as the fourth equation for the temperature Θ. The subsequent reduction of these four equations to a block of two equations corresponding to critical processes in the low-temperature layer at the front leads to a model capable of explaining the appearance of both the spiral waves observed in experiments and Turing structures, as was demonstrated in numerical two-dimensional simulations.

The model for radicals HO_2 and temperature Θ is as follows:

$$\frac{\partial u}{\partial t} = p - m\exp(-E/\Theta)u + D_u\Delta u, \tag{1}$$

$$\frac{\partial \Theta}{\partial t} = \gamma(m\exp(-E/\Theta)u - s(\Theta - \Theta_0)) + D_\Theta\Delta\Theta. \tag{2}$$

where the first term in the equation for the HO_2 radical corresponds to the rate of its production, the second term corresponds to the recombination of radicals with the activation energy E. The term $s(\Theta-\Theta_0)$ arises due to heat transfer across the layer and describes the temperature relaxation to the equilibrium value for the layer under consideration. The factor γ is responsible for the ratio of characteristic times. The Lewis number corresponds to the ratio of the diffusion coefficients D_Θ/D_u.

Numerical simulations of the model in a two-dimensional region demonstrate that, by varying the experimental conditions causing changes in the model parameters, it can explain the formation of both wave and cellular (Turing) structures at the combustion front.

The authors were supported by the Russian Science Foundation (project no. 22-21-00546).

REFERENCES

1. *Walgraef D.*, Spatio-temporal pattern formation: with examples from physics, chemistry, and materials science. Springer Science & Business Media, (2012).
2. *Ju Y., Maruta, K.*, “Microscale combustion: Technology development and fundamental research”, Progress In Energy And Combustion Science, **37**, 669–715 (2011).
3. *Yakupov E. O., Gubernov V. V., Polezhaev A. A.*, “Modeling of wave patterns at the combustion front”, Izvestiya VUZ. Applied Nonlinear Dynamics, **29**, No. 4, 538–548 (2021).
4. *Yakupov E. O., Gubernov V. V., Polezhaev A. A.*, “Theoretical Study of the Formation of Inhomogeneous Structures at the Combustion Front Based on the Reduced Model”, Bulletin of the Lebedev Physics Institute, **49**, No. 10, 329–335 (2022).

EXPERIMENTAL MODELING OF ENSEMBLES OF NEURON-LIKE OSCILLATORS

V. I. Ponomarenko[1], **M. D. Prokhorov**[2]

[1] *Saratov Branch of Kotel'nikov Institute of Radio Engineering and Electronics of Russian Academy of Sciences, Saratov, Russia;*
ponomarenkovi@gmail.com

[2] *Saratov Branch of Kotel'nikov Institute of Radio Engineering and Electronics of Russian Academy of Sciences, Saratov, Russia;*
mdprokhorov@yandex.ru

The development of information processing devices based on neuron-like oscillators is one of the main trends in modern interdisciplinary neuroscience [1]. Neural networks built from such neuron-like oscillators are spiking, or impulse neural networks (SNN) and effectively model the data processing and storage information in the brain [2]. Spiking neural networks have been successfully applied in practice for automatic processing of sound and visual information, detection of characteristic patterns in biomedical signals, and for solving information processing problems related to robot motion control.

Often the purpose of experimental modeling is to repeat the theoretically or numerically obtained results, which gives them the status of the real phenomena. In some cases, the study of ensembles containing a small number of coupled neuron-like oscillators is of interest. Such ensembles are able to solve adaptive problems of movement, pattern recognition, and their study can shed light on the essence of phenomena occurring in the brain [3].

Thus, the development of single neuron-like oscillators is a demanded task of neurodynamics. When developing such oscillators, one can be guided by the considerations of creating oscillators whose dynamics maximally corresponds to the differential equation of the neuron model, or development the simplest device that demonstrates qualitatively similar behavior.

Within the framework of this work, the principle of an electrical circuit construction of FitzHugh-Nagumo and Hindmarsh-Rose oscillators, built from the maximal correspondence to a differential equation, is considered. A simpler circuit of the FitzHugh-Nagumo oscillator has also been proposed, demonstrating qualitatively similar dynamics.

The dynamics of ensembles of neuron-like oscillators has been experimentally investigated.

This study was supported by the Russian Science Foundation, Grant No. 22-22-00150, https://rscf.ru/en/project/22-22-00150/.

REFERENCES

1. *Zhu J., Zhang T., Yang Y., Huang. R.* "A comprehensive review on emerging artificial neuromorphic devices," Applied Physics Reviews, **7**, No. 1, 011312 (2020).
2. *Sawicki J., et al.* "Perspectives on adaptive dynamical systems," arXiv:2303.01459, (2023).
3. *Kurkin S. A., Kulminskiy D. D., Ponomarenko V. I., Prokhorov M. D., Astakhov S. V., and Hramov A. E.* "Central pattern generator based on self-sustained oscillator coupled to a chain of oscillatory circuits" Chaos, **32**, 033117 (2022).

THE CHAOS GAME WITH ARBITRARY JUMP IN AN EXTENDED HYPERBOLIC PLANE

L. N. Romakina[1], I. V. Ushakov[2]

[1]*Saratov State University, Saratov, Russia;* romakinaln@mail.ru
[2]*Saratov State University, Saratov, Russia;* ivan.ushakov.99@ya.ru

We continue to study the dynamical system "Chaos game" in an extended hyperbolic plane H^2 [1, 2]. To organize the Chaos game with an arbitrary jump and various strategies on the polygons of the plane H^2, we derive formulae for dividing line segment in a given ratio λ. We find such formulae for segments of each of the three types (elliptic, hyperbolic, and parabolic). According to the new formulae, we expand the capabilities of the software package *pyv* [3] and use it to organize the Chaos game on various polygons. By experimentally choosing the most appropriate values for each particular polygon, we get clear frames of the game, without overlapping domains of points. Comparing the results obtained with similar results of the game in the Euclidean plane, we try to identify the properties of the dynamic system under study that do not depend on the metric of the plane of the game implementation.

REFERENCES

1. *Romakina L. N., Ushakov I. V.*, "Chaos game in an extended hyperbolic plane," Theoret. and Math. Phys., **215**, No. 3, 793–804 (2023).

2. *Romakina L. N., Ushakov I. V.*, “The Chaos game in the hyperbolic plane of positive curvature,” in: Nonlinear Dynamics & Integrability (NDI-2022), Abstracts (Satellite International Conference on Nonlinear Dynamics & Integrability and Scientific School “Nonlinear Days,” Yaroslavl, June 27 – July 1, 2022), Yaroslavl State Univ., Yaroslavl, 2022, pp. 86–88.
3. https://github.com/MrReDoX/pyv.

CONSTRUCTION OF PERIODIC SOLUTIONS OF A MATHEMATICAL MODEL OF A ROTATING DISK

V. D. Romanov

P.G. Demidov Yaroslavl State University, Yaroslavl, Russia;
romanovslava123@gmail.com

We study an initial boundary value problem

$$\upsilon_{tt} + a_0 a(|\upsilon_t|^2)\upsilon_t + (\mathrm{I} - \mu\Gamma(i\Omega))(b(|\upsilon_{xx}|^2)\upsilon_{xx})_{xx} = 0, \quad (1)$$

$$\upsilon|_{x=0} = \upsilon_x|_{x=0} = 0, \quad \upsilon|_{x=1} = \upsilon_x|_{x=1} = 0,$$

$$\upsilon|_{x=l_1-0} = \upsilon|_{x=l_1+0}, \quad \upsilon_x|_{x=l_1-0} = \upsilon_x|_{x=l_1+0},$$

$$(\mathrm{I} - \mu\Gamma(i\Omega))b(|\upsilon_{xx}|^2)\upsilon_{xx}|_{x=l_1+0}^{x=l_1-0} = -MJ\upsilon_{xtt}|_{x=l_1} - a_0 S_1 a(|\upsilon_{xt}|^2)\upsilon_{xt}|_{x=l_1},$$

$$(\mathrm{I}-\mu\Gamma(i\Omega))(b(|\upsilon_{xx}|^2)\upsilon_{xx})_x|_{x=l_1+0}^{x=l_1-0} = M\upsilon_{tt}|_{x=l_1} + a_0 S_2 a(|\upsilon_t|^2)\upsilon_t|_{x=l_1}, \quad (2)$$

$$\upsilon(x,t+s)|_{t=0} = \upsilon_0(x,s) \in D(\bar{Q}_-), \ \upsilon_1(x,0) = \dot{\upsilon}_0(x) \in H^2(0,1), \quad (3)$$

which has $0 \le x \le 1,\ 0 < l_1 < 1,\ t \ge 0,\ \upsilon(x,t) = y(x,t) + iz(x,t)$ $(i = \sqrt{-1})$, $a(\xi) = 1 + a_1\xi + ..,\ \ b(\xi) = 1 + b_1\xi + ...$,

$$(\mathrm{I} - \mu\Gamma(i\Omega))\varepsilon(t) \equiv \varepsilon(t) - \mu \int\limits_{-\infty}^{0} R(s)\mathrm{e}^{i\Omega s}\varepsilon(t+s)ds,$$

$0 < \mu < 1,\ \Omega \ge 0,\ \ M, J, a_0, S_1, S_2$ - positive parameters, $D(\bar{Q}_-)$ and $H^2(0,1)$ function spaces of initial conditions.

The initial boundary value problem (1) - (3) is a mathematical model of a rotating solid disk on a flexible shaft, which material is nonlinearly hereditarily viscoelastic [1]. The equation (1) is obtained in [2]. The mathematical model is given in dimensionless variables. The dimensionless parameters characterize: Ω - angular velocity of disk rotation, a_0 - external friction coefficient, M - disk mass, J - disk moment of inertia, S_1, S_2 - disk geometric properties, l_1 - disk attachment point.

We study the conditions of occurrence and the type of auto-oscillatory solutions arising in the initial boundary value problem (1) - (3) under the change of parameters and due to the loss of stability of the zero state.

REFERENCES

1. *Rabotnov Y. N.*, "Elements of hereditary solid mechanics" in : Science, Moscow, 1977, 384 p.
2. *Kubyshkin E. P.*, "Some problems of the dynamics of distributed rotors", Yaroslavl: Yaroslavskij Gosudarstvennyj Universitet Im. P. G. Demidova. Matematicheskij Fakultet, 157–181 (2001).

ON THE PROBLEM 1100 AS A WHOLE

D. W. Serow

State Institute of Economy Finance Law and Technology, 188300 Military Glory City Gatchina, Russia; dimusum@yandex.ru

In 2021 author has published some examples of nonwandering continua possessing the Wada property. At the same time, a nonwandering continuum has been constructed by itself art turning out to be common boundary two regions is constituting to be Birkhoff curve [1]. The analytic construction problem for a dynamic system possessing Birkhoff curve property was turned out to be the problem 1100 essentence [2]. The problem 1100 solution allows to study these dynamic systems.

The maps possessing the Birkhoff curve property on iterations are defined to be analytically into two qualitative different variants as follows.

$$\frac{(R\tilde{x} + \tilde{y} + ipq\tilde{y})^2}{p^2\varrho} \mathrm{e}^{i\Omega(\tilde{x},\tilde{y})} \mapsto x + iy, \quad q = p = \pi/2, \tag{1}$$

and

$$\frac{(\varrho + \varrho_0)\,(R\tilde{x} + \tilde{y} + ipq\tilde{y})^2}{p^2\varrho^2} \mathrm{e}^{i\Omega(\tilde{x},\tilde{y})} \mapsto x + iy, \quad q, p > 1.92, \ \varrho_0 > 0 \tag{2}$$

where $(\tilde{x}, \tilde{y}) \stackrel{def}{=} (\tanh \beta x, \tanh \beta y)$ and further

$$\varrho \stackrel{def}{=} \frac{\sqrt{(R\tilde{x} + \tilde{y})^2 + (pq\tilde{y})^2}}{p} \quad \text{and} \quad \Omega(\tilde{x}, \tilde{y}) \stackrel{def}{=} R\frac{5\pi(\tilde{y}^2 + \alpha\tilde{x}^2)}{3} + \Delta$$

with other parameters $R = \pm 1,\ \alpha = 0.23,\ \beta = 1.1\ \Delta = \pi R/45$.

Thus there subsists a chance to take by the udder a woolly rhinoceros female.

Theorem 1. *Birkhoff curve contains degenerate fixed point.*

The degenerate fixed point index is equal zero.

Theorem 2. *There exist two different topologically types Birkhoff curves turning out to be structure stable and structure unstable.*

Theorem 3. *Number of degenerate fixed point delivers topologically differences as for structure stable well as for structure unstable Birkhoff curves.*

Open Conjecture: *somewhere the Petersen snark is hiding.*

If this conjecture is not turning out to be phantom, then one can to take by the udder the Birkhoff curves problem. In any case, it is possible to construct the snarks sequence, with first from one is turns out to be the Petersen snark.

REFERENCES

1. *Serow, D. W.*, "Nonwandering Continuum Possessing the Wada Property," Theor. and Math. Phis., **207**, No. 3, 841–853 (2021).
2. *Strein, S. van,* "One-dimensional versus two-dimensional dynamics". In book: "Open Problems in Topology", Ed. by Jan van Mill & George M. Reed, Elsevier Sc. Pub. B.V. (North-Holland), 646–654 (1990).

SEMICLASSICAL ASYMPTOTIC SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH A DELTA-POTENTIAL AND RAPIDLY OSCILLATING INITIAL CONDITIONS

A. I. Shafarevich[1], **O. A. Shchegortsova**[2]

[1]*Lomonosov Moscow State University, Russia;* shafarev@yahoo.com
[2]*Lomonosov Moscow State University, Russia;*
olga.shchegortsov@gmail.com

We consider the Schrödinger equation with a delta-potential localized on a surface of codimension 1. The Schrödinger operator with a delta-potential is defined as a self-adjoint extension of the Schrödinger operator with a smooth potential restricted to the functions vanishing on the surface of the carrier of the delta-function. The domain of a such an operator consists of functions satisfying boundary conditions on the carrier of the delta-potential.

In thiw work, we describe asymptotics as $h \to 0+$ of the solution of the Cachy problem with initial conditions representing a rapidly oscillating wave packets. Such solutions are called narrow Gaussian beams. We provide the solution in terms of the Maslov complex germ theory. Semiclassical asymptotics can be associated with invariant geometrical objects of corresponding classical Hamiltonian systems. The form of the delta-potential defines a mapping between these objects.

The work is supported by the grant of the Theoretical Physics and Mathematics Advancement Foundation "BASIS".

REFERENCES

1. *Maslov V. P.,* The Complex WKB Method for Nonlinear Equations, Nauka, Moscow, 1977.
2. *Maslov V. P.,* Asymptotic methods and perturbation theory, Nauka, Moscow, 1988.
3. *Shafarevich A. I., Shchegortsova O. A.* Maslov complex germ and semiclassical contracted states in the Cauchy problem for the Schrödinger equation with delta-potential, Contemporary Mathematics. Fundamental Directions, Vol. 68, pp. 704–715.

ARGUMENT SHIFT METHOD IN $U\mathfrak{gl}_d$

G. I. Sharygin

Lomonosov Moscow State University, Moscow, Russia;
ITEP, Moscow, Russia;
Sino-Russian Mathematics Center, PKU, Beijing, China;
sharygin@itep.ru

Let X be a Poisson manifold. Argument shift method is a well-known method of obtaining commutative subalgebras in the Poisson algebra $C^\infty(X)$. It is based on the observation that if a vector field ξ and the Poisson bivector π verify the following equalities

$$\mathcal{L}_\xi \pi \neq 0,\ \mathcal{L}_\xi^2 \pi = 0,$$

where $\mathcal{L}_\xi$ denotes the Lie derivative in direction of ξ, then the elements $\mathcal{L}_\xi^p(f)$ Poisson-commute with each other for all $p \geq 0$ and all Casimir functions f.

An important particular case of this construction is when $X = \mathfrak{g}^*$ is equal to the dual space of a Lie algebra and the Poisson structure is equal to the canonical Lie-Poisson structure on $\mathfrak{g}^*$ (i.e. is induced from the Lie algebra structure on $\mathfrak{g}$) and the vector field is constant with respect to affine coordinates on $\mathfrak{g}^*$. This situation was first considered by Mischenko and Fomenko in 1. In this case for a generic vector field ξ the method gives a maximal commutative subalgebra in the symmetric algebra $S(\mathfrak{g})$, often called *Mischenko-Fomenko algebra*.

In 1991 E. B. Vinberg asked the following question: *Is it possible to find a commutative subalgebra A_ξ in the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra, such that its image in $S\mathfrak{g}$ under the canonical isomorphism of associated graded algebra of $U\mathfrak{g}$ and $S\mathfrak{g}$ would be equal to Mischenko-Fomenko algebra?* This question was solved by various people, the best known construction of the *quantum Mischenko-Fomenko algebra A_ξ* being that of Rybnikov, see 2. However, finding an element in Rybnikov's algebra that will correspond to a particular $\xi^p(f)$ in Mischenko-Fomenko algebra.

In my talk I will describe a method of quantising the element $\xi^p(f)$ (i.e. raising it to $A_\xi \subset U\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{gl}_d$. It is based on a systematic use of the *quasiderivation operation* of Gurevich and Saponov, (see 3.) $\hat{\xi}$ on $U\mathfrak{gl}_d$ in the stead of usual directional derivative ξ (we assume that the coefficients of $\hat{\xi}$ coincide with those of ξ). Namely, we can prove the following result

Theorem. *Let $\hat{f} \in U\mathfrak{gl}_d$ be a central element, $p \geq 0$; then the element $\hat{\xi}^p(\hat{f})$ is in the quantum Mischenko-Fomenko algebra A_ξ. In particular, all such elements commute with each other.*

The talk is based on a joint work with Yasushi Ikeda, `arXiv:2307.15952`.

REFERENCES

1. *Mishchenko A. S., Fomenko A. T.,* "Euler equations on finite-dimensional Lie groups," Math. USSR-Izv., **12**, No. 2, 371–389 (1978).
2. *Rybnikov L. G.,* "The argument shift method and the Gaudin model," Funct. Anal. Appl. **40** 188–199 (2006).
3. *Gurevich D., Pyatov P., Saponov P.,* "Braided Weyl algebras and differential calculus on U(u(2))," J. Geom. Phys., **62**, No. 5, 1175–1188 (2012)

IMPLEMENTATION OF A CHAOTIC OSCILLATION GENERATOR BASED ON THE CONCEPT OF MIXED DYNAMICS

D. S. Shchapin

Institute of Applied Physics RAS, Nizhny Novgorod, Russia;
`shapinds@mail.ru`

The papers [1,2] give a mathematical justification for the existence of a new third type of chaos — the so-called mixed dynamics, which is characterized by the fundamental inseparability of dissipative and conservative behavior. Three different forms of dynamic chaos are currently known: "dissipative chaos", "conservative chaos" and "mixed dynamics". This article provides an implementation of a chaotic mixed dynamics generator implemented on a field programmable gate array (FPGA) [3]. The generator model is based on the dynamics of two adaptively coupled Kuramoto phase generators [4-6]. It is experimentally shown that oscillations appear in the generator corresponding to a chaotic attractor and a chaotic repeller (attractor in reversed time). It has been established that for different values of the generator parameters, either ordinary dissipative chaos or mixed dynamics can be realized. It is shown that in the case of mixed dynamics, the behavior of trajectories in the phase space becomes more complex, and the spectral characteristics change towards a more uniform distribution of power over the frequencies of the spectrum. The influence of a harmonic external force on the mixed dynamics of the generator is also investigated and it is shown that this leads to an additional complication of the dynamic behavior of the generator output signals.

This work was supported by the Russian Science Foundation under Project No. 19-12-00338

REFERENCES

1. *Gonchenko S. V., Shilnikov L. P., Turaev D. V.,* "On newhouse domains of twodimensional diffeomorphisms which are close to a diffeomorphism with a structurally unstable heteroclinic cycle," Proc. steklov inst. math, **216**, 70–118 (1997).
2. *Gonchenko S. V., Turaev D. V.,* "On three types of dynamics and the notion of attractor," Proc. steklov inst. math, **297**, 116–137 (2017).
3. *Shchapin D. S., Emelianova A. A., Nekorkin V. I.,* "A chaotic oscillation generator based on mixed dynamics of adaptively coupled Kuramoto oscillators," Chaos, Solitons & Fractals, **166**, No. 3, 112989 (2023).
4. *Emelianova A. A., Nekorkin V. I.,* "On the intersection of a chaotic attractor and a haotic repeller in the system of two adaptively coupled phase oscillators," Chaos, **29**, No. 11, 111102 (2019).
5. *Emelianova A. A., Nekorkin V. I.,* "The third type of chaos in a system of two adaptively coupled phase oscillators," Chaos, **30**, No. 5, 051105 (2020).
6. *Emelianova A. A., Nekorkin V. I.,* "Emergence and synchronization of a reversible core in a system of forced adaptively coupled Kuramoto oscillators," Chaos, **31**, No. 3, 033102 (2021).

PARABOLIC HITCHIN SYSTEMS

O.K. Sheinman

Steklov Mathematical Institute, Moscow, Russia;
sheinman@mi-ras.ru

A class of parabolic Hitchin systems given an effective description via Separation of Variables will be presented.

ASYMPTOTICS OF THE SOLUTION OF A SINGULARLY PERTURBED SYSTEM OF EQUATIONS WITH A SINGLE-SCALE INTERNAL LAYER

R. E. Simakov

Lomonosov Moscow State University, Moscow, Russia;
simakov.re14@physics.msu.ru

Consider the boundary value problem for a system of equations

$$\varepsilon^2 \frac{d^2u}{dx^2} = F(u, v, x, \varepsilon), \quad \varepsilon \frac{d^2v}{dx^2} = f(u, v, x, \varepsilon), \quad 0 < x < 1$$

with Neumann boundary conditions, where $\varepsilon > 0$ is a small parameter, $u(x, \varepsilon)$ and $v(x, \varepsilon)$ are the unknown scalar functions.

A specific feature of the problem is that the equation $F(u, v, x, 0) = 0$ has a double root $u = \varphi(v, x)$, and the equation $f(\varphi(v, x), v, x, 0) = 0$ has three nonintersecting simple roots $v = \psi_i(x)$, $x \in [0; 1]$, $i = 1, 2, 3$.

We state the conditions under which for sufficiently small ε the problem has a solution $u(x, \varepsilon)$, $v(x, \varepsilon)$ that has a transition layer in a neighborhood of some interior point x_0 of the interval $(0; 1)$. In this neighborhood the solution makes a fast transition from values close to the one solution of the degenerate system $u = \varphi(\psi_1(x), x)$, $v = \psi_1(x)$ to values close to the other solution $u = \varphi(\psi_3(x), x)$, $v = \psi_3(x)$. In the limit $\varepsilon \to 0$ there is a "step" at point $x = x_0$. Such solutions are called *Step Type Contrast Structures* (STCS's).

An asymptotics of this STCS is constructed with respect to a small parameter ε. It qualitatively differs from the well-known expansion in the case where all the roots of the degenerate equations are simple [1] but also does not coincide with the expansions in the previously studied problems with double roots [2]; in particular, the inner transition layer turns out to be single-scale.

Justification of the asymptotics is made using the asymptotic method of differential inequalities, the application of which in this problem also has some specific features.

REFERENCES

1. *Butuzov V. F., Levashova N. T., Mel'nikova A. A.*, "Steplike contrast structure in a singularly perturbed system of equations with different powers of small parameter," Comput. Math. and Math. Phys, **52**, No. 11, 1526–1546 (2012).

2. *Butuzov V. F., Simakov R. E.*, “Asymptotics of the Solution of a Singularly Perturbed System of Equations with a Multizone Internal Layer,” Differential Equations, **57**, No. 4, 415–445 (2021).

QUANTIFYING CHAOS USING LAGRANGIAN DESCRIPTORS

Ch. Skokos

Nonlinear Dynamics and Chaos Group Department of Mathematics and Applied Mathematics, University of Cape Town Rondebosch, 7701, Cape Town, South Africa;

haris.skokos@gmail.com haris.skokos@uct.ac.za

We present simple and efficient methods to estimate the chaoticity of orbits in conservative dynamical systems, namely, autonomous Hamiltonian systems and area-preserving symplectic maps, from computations of Lagrangian descriptors (LDs) [1, 2] on short time scales. In particular, we consider methods based on the difference and the ratio of the LDs of neighboring orbits, as well as a quantity related to the finite-difference second spatial derivative of the LDs [3, 4]. We use these indices to determine the chaotic or regular nature of ensembles of orbits of the prototypical two degree of freedom Hénon–Heiles system, as well as the two- and the four-dimensional standard maps. We find that these indicators are able to correctly identify the chaotic or regular nature of orbits to better than 90% agreement with results obtained by the Smaller Alignment Index (SALI) method of chaos detection [5, 6]. In addition to quantifying chaos, the studied indicators can reveal details about the systems' local and global chaotic phase space structure. Our findings indicate the capability of LDs to efficiently identify chaos in conservative dynamical systems without knowing the variational equations (tangent map) of continuous (discrete) time systems needed by traditional chaos indicators.

REFERENCES

1. *Madrid J. J., Mancho A. M.*, “Distinguished trajectories in time dependent vector fields,” Chaos, **19**, 013111 (2009).
2. *Mancho A. M., Wiggins S., Curbelo J., Mendoza C.*, “Lagrangian descriptors: A method for revealing phase space structures of general time dependent dynamical systems,” Communications in Nonlinear Science and Numerical Simulation, **18**, 3530–3557 (2013).

3. *Hillebrand M., Zimper S., Ngapasare A., Katsanikas M., Wiggins S., Skokos Ch.*, "Quantifying chaos using Lagrangian descriptors," Chaos, **32**, 123122 (2022).
4. *Zimper S., Ngapasare A., Hillebrand M., Katsanikas M., Wiggins S., Skokos Ch.*, "Performance of chaos diagnostics based on Lagrangian descriptors. Application to the 4D standard map," Physica D, in press, ArXiv: 2305.00978 (2023).
5. *Skokos Ch.*, "Alignment indices: A new, simple method for determining the ordered or chaotic nature of orbits," J. Phys. A, **34**, 10029–10043 (2001).
6. *Skokos Ch., Manos T.*, "The Smaller (SALI) and the Generalized (GALI) alignment indices: Efficient methods of chaos detection," Lect. Notes Phys., **915**, 129–181 (2016).

DUBROVIN'S METHOD FOR MKDV HIERARCHY

A. O. Smirnov[1], **I. V. Anisimov**[2]

[1]*St.-Petersburg State University of Aerospace Instrumentation, St.-Petersburg, Russia;* alsmir@guap.ru

[2]*St.-Petersburg State University of Aerospace Instrumentation, St.-Petersburg, Russia;* igorek.anisimov.2017@mail.ru

Let us consider the Lax operator

$$\Psi_x = U\Psi,$$

where $U = \lambda J + Q$,

$$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix},$$

$v \in \mathbb{R}$, $i^2 = -1$.

In the case of a finite-gap potential, there is a matrix function $\widehat{\Psi} = M\Psi$ that satisfies the same Lax equation [1]. Here M is a monodromy matrix, which is a polynomial in the spectral parameter λ [1].

From the compatibility condition we get that the matrix M has the following structure

$$M = V_n + \sum_{k=1}^{n-1} V_{n-k} + c_n U + c_{n+1} J,$$

where $V_1 = \lambda U + V_1^0$, $V_{k+1} = \lambda V_k + V_{k+1}^0$,

$$V_k^0 = \begin{pmatrix} F_k & H_k \\ G_k & -F_k \end{pmatrix}, \quad k \geq 1.$$

The elements of the matrices V_k^0 satisfy the following recurrence relations

$$H_1 = \frac{i}{2} v_x, \quad G_1 = -\frac{i}{2} v_x, \quad \partial_x F_k = (G_k - H_k) v,$$
$$H_{k+1} = \frac{i}{2} \partial_x H_k + i v F_k, \quad G_{k+1} = -\frac{i}{2} \partial_x G_k + i v F_k.$$

Using recurrent relations, it is not difficult to obtain the following reality conditions

$$G_{2k-1} = H_{2k-1}^* = -H_{2k-1},$$
$$G_{2k} = H_{2k}^* = H_{2k},$$
$$F_{2k-1} \in i\mathbb{R}, \quad F_{2k} = 0.$$

Let the second equation of the Lax pair have the form

$$\Psi_{t_k} = W_k \Psi,$$

where $W_k = 2^{2k} V_{2k}$. Then from compatibility conditons we get the following integrable nonlinear equations

$$v_{t_k} = 2^{2k} \partial_x H_{2k}.$$

In particular,

$$v_{t_1} = 6v^2 v_x - v_{xxx},$$
$$v_{t_2} = 30 v^4 v_x - 10 v_x^3 - 40 v v_x v_{xx} - 10 v^2 v_{xxx} + v_{xxxxx}.$$

Therefore, the integrable nonlinear equations we have constructed are equations from the mKdV hierarchy.

It is well known that the Miura transformation (see for ex. [2])

$$u_\pm = -v^2 \mp v_x.$$

links the solutions of the mKdV equation with the solutions of the KdV equation

$$u_{t_1} + u_{xxx} + 6 u u_x = 0.$$

The spectral curve of the multiphase solution of the mKdV equation is also hyperelliptic, since it is determined by the characteristic equation of the monodromy matrix

$$\det(M - i\mu I) = 0,$$

or

$$\mu^2 = \det M = \lambda^{2n+2} + \sum_{j=1}^{2n+2} f_j \lambda^{2n+2-j}.$$

Here I is identity matrix, and f_j are integrals of the hierarchy equations. It follows from the equation of the spectral curve that its genus is equal to n if all the roots of the polynomial $\det M$ are simple.

In our work, we constructed the simplest nontrivial solutions of the mKdV equation and analyzed the connections between the spectral curves of solutions linked by the Miura transformation.

The work has been supported by the Russian Science Foundation (grant agreement No 22-11-00196).

REFERENCES

1. *Dubrovin B. A.*, "Matrix finite-zone operators," J. Sov. Math., **28**, No. 1, 20–50 (1985).
2. *Calogero F., Degasperis A.*, Spectral transform and solitons: tools to solve and investigate nonlinear evolution equations, North-Holland Publ. Comp., Amsterdam, New York, Oxford (1982).

DYNAMICS OF OSCILLATOR POPULATIONS GLOBALLY COUPLED WITH DISTRIBUTED PHASE SHIFTS

L. A. Smirnov[1], **A. S. Pikovsky**[2]

[1]*Department of Control Theory, Nizhny Novgorod State University, Gagarin Av. 23, 603022, Nizhny Novgorod, Russia;*
smirnov.lev.al@gmail.com

[2]*Department of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Str. 24/25, 14476, Potsdam-Golm, Germany*

Globally coupled populations of oscillators are paradigmatic models for synchronization and processes of the emergence of collective modes. In many cases, the global nature of coupling is determined by the setup. However, the oscillators may have different properties and possess intrinsic noise, resulting in non-identity of the system elements. The most common source of non-identity is the variation in natural frequencies among the oscillators. Such a feature has been already studied in detail. In our research, we specifically investigate the impact of disorder in the phase shifts of the coupling. These phase lags naturally appear where the global force has to propagate to reach spatially distributed elements belonging to the population.

First, we derive a phase model in the Kuramoto – Sakaguchi form for an ensemble of quadratic integrate-and-fire neurons recurrently coupled together via a synaptic current taken with different time shifts. From a mathematical point of view, in such a situation, the global force acts on different oscillators with different time delays, and these delays respectively result in a spread of the phase lags in the corresponding phase model. In order to demonstrate this, we assume a weak interaction between system units, then transform to slow varying phases, and use the time-average procedure.

Second, we show that a distribution of the phase shifts in coupling can also occur from a distribution of the local properties of oscillators. To this end, we consider the canonical θ-neuron model. Here, as in the previous case, we suppose neurons interact all-to-all via chemical synapses, but we take into account that the corresponding synaptic current forcing on the neuron satisfies the relaxation equation and has an individual (disordered) value of the relaxation constant. For this situation, assuming weak coupling and employing multiple time scale analysis, one can also obtain the Kuramoto – Sakaguchi model of phase oscillators with distributed phase lags.

Next, we analyze the properties of the phase model. In the thermodynamics limit, the continuum of phase oscillators can be characterized by the one-particle probability density function, which evolves according to the continuity equation, possessing an exact solution in the form of the Ott – Antonsen ansatz at every value of the phase lags α. This manifold is attracting (as shown in other papers) and corresponds to a special ansatz for the expansion of in a Fourier series with respect to phase variable in the form of a Poisson kernel. Using such an analytical approach, we get the low-dimensional description for the collective behavior of the corresponding system, where in the equations for macroscopic complex fields only the redefined order parameter $Q(t, \alpha)$ depends on the phase shifts α through initial conditions, but its dynamics are not. Based on these reduced equations and employing stability analyses in the linear approximation, we provide the arguments that in course of the dynamics the memory about the initial state gets lost and $Q(t,\alpha) \to Q(t)$. After the convergence to $Q(t)$, the dynamics of the population with a distribution of phase shifts reduces to a single dynamical equation for the auxiliary order parameter $Q(t)$, and the original order parameters are related to it via circular moments of the distribution $f(\alpha)$ of the phase shifts α.

All theoretical concepts are confirmed by numerical calculations performed directly within the considered models of oscillator populations.

This work was supported by the Russian Science Foundation (project no. 22-12-00348).

MULTIVALUED QUANDLES AND ASSOCIATED BIALGEBRAS

D. V. Talalaev[1,2]

[1]*M.V. Lomonosov State University, Moscow, Russia;*
`dtalalaev@yandex.ru`
[2]*P.G. Demidov State University, Yaroslavl, Russia;*

Multivalued groups arise in a fairly wide field of mathematical subjects: representation theory [1], discrete dynamical systems [2], algebraic geometry. The report will talk about a similar construction in the category of quandles, structures that have significant applications in knot theory and descrete integrable models. I will introduce the concept of an n-quandle, give some examples, describe the coset construction in this case, talk about a natural analogue of a group algebra for a quandle, and construct an n-corack bialgebra associated with a multivalued quandle. The report is based on the results of joint work with V. Bardakov and T. Kozlovskaya in preparation.

REFERENCES

1. *Buchstaber V.M., Rees, E. G.* "Multivalued groups, their representations and Hopf algebras," Transform. Groups2(1997), no.4, 325-349.
2. *Buchstaber, V.Ṁ., Veselov, A.Ṗ.* "Integrable correspondences and algebraic representations of multivalued groups," Internat. Math. Res. Notices (1996), No.8, 381-400.

CONSTRUCTION OF QUASINORMAL FORMS FOR SYSTEMS OF TWO EQUATIONS WITH LARGE DELAY

A. O. Tolbey

Regional Scientific and Educational Mathematical Center of the Yaroslavl State University, Yaroslavl, Russia;
a.tolbey@uniyar.ac.ru

The local dynamics of systems of two equations with delay is considered. The main assumption is that the delay parameter is large enough. Critical cases are singled out in the problem of the stability of the equilibrium state and it is shown that they have infinite dimension. Research methods based on the application of the theory of invariant integral manifolds and normal forms turn out to be not directly applicable. The main results include the construction of special nonlinear boundary value problems which play the role of normal forms. Their nonlocal dynamics determines the behaviour of all solutions of the original system in the vicinity of the equilibrium state.

The study was supported by the Russian Science Foundation, grant no. 21-71-30011, https://rscf.ru/en/project/21-71-30011/

FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STRETCHING, ROTATION AND SHIFTS

A. A. Tovsultanov

Kadyrov Chechen State University, Grozny, Russia;
a.tovsultanov@mail.ru

In the first part of the report, the Dirichlet problem in a flat bounded domain will be considered for a second-order functional differential equation containing in the highest derivatives a combination of transformations of arguments $x \mapsto x/p$ $(p > 0)$ and $x \mapsto -x$ to the desired function. Necessary and sufficient conditions are found in the algebraic form for the fulfillment of a Gording-type inequality, which ensures unique (Fredholm) solvability, discreteness and sectorial structure of the spectrum of the Dirichlet problem. In the literature, the term strongly elliptic equation is adopted in this situation. The derivation of the mentioned conditions, expressed directly through the coefficients of the equation, is based on a combination of Fourier and Gelfand transformations of elements of the commutative B^*-algebra of bounded operators in $L_2(\mathbb{R}^2)$, generated by the operators of dilation, symmetry with respect to the origin and multiplication by continuous ones into homogeneous functions of degree zero in $\mathbb{R}^2 - \{0\}$. The main point here is to clarify the structure of the space of maximal ideals of this algebra. It is proved that the space of maximal ideals is homeomorphic to the direct product of the spectrum of the dilation operator (circle), the projective line and the two-point set $\{\pm 1\}$. The algebraic conditions found ensure that the equation is strongly elliptic for any flat bounded domain. If the closure of the region contains the origin of coordinates, and orbits of any length fall inside the region, the mentioned algebraic conditions turn out to be necessary for strong ellipticity. The proof is based on the matrix approach, widely used in the theory of differential-difference operators.

At present, boundary value problems both for elliptic differential-difference equations and for equations with compression and expansion of a multidimensional independent variable have been studied quite well. The influence of a combination of shifts and compression of the argument in the higher derivatives of the desired function on the solvability of the boundary value problem has not previously been considered in the mathematical literature, both domestic and foreign. Difficulties that arise when studying elliptic functional differential equations with affine transformations are due to the fact that compression and shift are non-commuting transformations, and the presence of both compression and shift in the equation leads to the appearance of an infinite number of fixed points for transformations of arguments. On the other hand, the case under consideration differs from equations where there is only a shift of the argument in that it allows a wide class of bounded domains that are invariant under the corresponding coordinate transformations. In order to take these features into account, a new approach to defining a functional operator with shifts and contractions of arguments is used: the study of such operators is proposed to be combined into a general scheme based on the corresponding integral over a regular Borel measure. For a functional operator with an affine transformation of an argument, defined as a composition of an argument compression operator (the function on which the operator acts) and a convolution operator with a regular Borel measure concentrated on a compact set, a formula for the spectral radius is derived (in the space $L_2(\mathbb{R}^2)$) based on the characteristic function of the measure. Explicit expressions for spectral radius in the case of atomic measures, as well as a number of other measures. The action of a functional operator with affine transformations on functions defined in a bounded domain is studied. The proposed approach is then applied to a model boundary value problem for an elliptic operator containing an affine transformation of the argument. A sufficient condition for unique solvability is established Dirichlet problem It is shown that if such a condition is violated in a boundary value problem, the correctness property may be lost.

The work was carried out within the framework of the state assignment of the Ministry of Education and Science of the Russian Federation under the project: Nonlinear singular integro-differential equations and boundary value problems (FEGS-2020-0001).

NORMALISATION FLOW

D. V. Treschev

Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, Russia; treschev@mi-ras.ru

The normalisation process in the theory of normal forms traditionally occurs step by step: unwanted terms (in a vector field, Hamilton functions, etc.) are removed one by one degree by degree. I will indicate a differential equation in the space of all Hamiltonians with a singular point at the origin, along the solutions of which the Hamiltonian functions move to their normal forms. Shifts along the flow of this equation correspond to canonical coordinate transformations. Thus, we are talking about a continuous normalisation procedure. The formal aspect of the theory does not cause difficulties. The analytical aspect and the convergence of the series are, as always, nontrivial. Only the first steps have been taken in this direction.

THE STRUCTURE OF THE INTERNAL TRANSITION LAYER IN THE REACTION-DIFFUSION PROBLEM IN THE CASE OF A BALANCED REACTION WITH A WEAK DISCONTINUITY

V. T. Volkov[1], **E. I. Nikulin**[2]

[1]*Moscow Lomonosov State University, Department of Physics, Moscow, Russia;* volkovvt@mail.ru

[2]*Moscow Lomonosov State University, Department of Physics, Moscow, Russia;* nikulin@physics.msu.ru

One of the actual problems of the theory of singular perturbations at present is the study of nonlinear singularly perturbed partial differential equations, the solutions of which have boundary and inner layers. Such equations are of great interest both in the qualitative theory of differential equations and in many applied problems. In particular, in mathematical models of chemical kinetics, synergetics, nonlinear wave theory, biophysics and other fields of physics, where the processes under study are described by nonlinear parabolic equations with small parameters at derivatives. Solutions to such problems may contain narrow areas of rapid parameter change: boundary or internal transition layers (contrast structures) of various types – stationary or moving fronts [1].

Reaction-diffusion and reaction-diffusion-advection equations are also intensively studied due to the fact that they act as mathematical models that reveal the main mechanisms that determine the behavior of more complex physical systems. In particular, the well-known FitzHugh-Nagumo system in the stationary case can be reduced to the problem considered in the article. Also, the system of equations of the drift-diffusion model of a semiconductor with an N-shaped dependence of the drift velocity on the electric field strength is reduced to the problem posed below.

The reason for the formation of transition layers (contrasting structures) in singularly perturbed reaction-diffusion-advection models can be the fulfillment of the reaction balance condition at some point or on some curve lying in the field of consideration or advection balance, as well as the gap of coefficients by spatial coordinate [2–5].

In the works [2, 3] for the case of continuous coefficients, the so-called *critical case* was considered when the reaction balance condition is fulfilled identically, i.e. at any point in the domain. In this paper, we consider a boundary value problem for a nonlinear singularly perturbed reaction-diffusion equation in the critical case in the presence of a weak discontinuity of the reactive term. By *weak discontinuity* is meant a discontinuity of the first kind at some point, which the function of sources undergoes in the first order with respect to a small parameter.

The following problem is considered:

$$\begin{cases} N_\varepsilon u := \varepsilon^2 \dfrac{\partial^2 u}{\partial x^2} - f(u,x) - \varepsilon f_1(u,x) = 0, \quad -1 < x < 1, \\ f_1(u,x) := \begin{cases} f_1^{(+)}(u,x), & u \in I_u, \quad x > x_p \\ f_1^{(-)}(u,x), & u \in I_u, \quad x < x_p \end{cases} \\ \dfrac{\partial u}{\partial x}(-1,\varepsilon) = 0, \quad \dfrac{\partial u}{\partial x}(1,\varepsilon) = 0 \,. \end{cases} \tag{1}$$

Here $0 < \varepsilon < \varepsilon_0 \ll 1$ – small parameter, $x_p \in (-1;1)$, I_u - the function $u(x,\varepsilon)$ change segment.

For this singularly perturbed reaction-diffusion equation the structure of the internal transition layer is investigated in the case of a balanced reaction with a weak discontinuity. It is shown that in the case of the balanced reaction, the presence of even a weak (asymptotically small) reaction discontinue can lead to the formation of several contrast structures of finite size, which may be stable or unstable.

The conditions under which there is a solution of the contrast structure type having an internal transition layer localized in the vicinity of the reaction break point are formulated. The existence of solutions with an internal transition layer (contrast structures) is proved, the question of their stability is investigated, and an asymptotic approximation with respect to a small parameter of this solutions is constructed. Sufficient conditions are formulated that determine either the asymptotic Lyapunov stability or the instability of each such solution.

The asymptotic approximation is constructed according to the method [6]; the asymptotic method of differential inequalities is used to prove the existence of the solution [7], as well as the asymptotic method of differential inequalities developed for problems with discontinuous nonlinearities [4, 5]; the study of stability is carried out by the method of tapering barriers [2].

The authors were supported by the Russian Science Foundation (project no. 23-11-00069).

REFERENCES

1. *Nefedov N. N.* "Development of methods of asymptotic analysis of transition layers in reaction-diffusion-advection equations:theory and applications", Computational Mathematics and Mathematical Physics, **61**, No. 12, 2068–2087, (2021).
2. *Volkov V. T., Nefedov N. N.* "Development of the asymptotic method of differential inequalities for investigation of periodic contrast structures", Computational Mathematics and Mathematical Physics, **46**, No. 4, 585–593, (2006).
3. *Nefedov N. N., Recke L., Schneider K. R.* "Existence and asymptotic stability of periodic solutions with an interior layer of reaction-advection-diffusion equations", Journal of Mathematical Analysis and Applications, **405**, 90–103, (2013).
4. *Nefedov N. N., Nikulin E. I., Orlov A. O.* "Existence of contrast structures in a problem with discontinuous reaction and advection", Russian Journal of Mathematical Physics, **29**, No. 2, 214–224, (2022).
5. *Nefedov N. N., Nikulin E. I., Orlov A. O.* "Contrast structures in the reaction-diffusion-advection problem in the case of a weak reaction discontinuity", Russian Journal of Mathematical Physics, **29**, No. 1, 81–90, (2022).
6. *Vasil'eva A. B., Butuzov V. F., Nefedov N. N.* "Contrast structures in singularly perturbed problems", Fund. and applied mathematics, **4**, 799–851, (1998) (in russian).
7. *Nefedov N. N.* "Method of differential inequalities for some classes of nonlinear singularly perturbed problems with internal layers", Differential Equations, **31**, No 7, 1132–1139, (1995).

DIFFERENTIAL EQUATION OF THE NEUTRAL DELAY TYPE WITH A DISCONTINUOUS NONLINEARITY

J. A. Yakubiv

P. G. Demidov Yaroslavl State University, Yaroslavl, Russia;
`yuliyayakubiv1004t@gmail.com`

Consider the equation

$$\dot{x} + x = \alpha \cdot \operatorname{sign}(\dot{x}(t-T)), \tag{1}$$

where α is a nonzero constant and let S_h be the class of initial conditions defined as

$$S_h = \{\varphi(t) \in C^1[-T, 0] \ : \ \dot{\varphi}(t) > 0, \ \varphi(0) = h\}, \tag{2}$$

where h is an arbitrary parameter. The objective of this work is to study the dynamics of this equation and to find stable periodic solutions.

Constructing the solutions of the equation (1) explicitly we derive these three non-overlapping cases:

$$\text{a) } \alpha > h;$$

$$\text{b) } \frac{h}{1-2e^T} < \alpha < h;$$

$$\text{c) } \alpha < h < (1-2e^T)\alpha.$$

If $\alpha < 0$, then by choosing an appropriate h we can achieve all three cases. However, if $\alpha > 0$, then only cases (a) and (b) coexist.

Theorem. *The solution of the equation (1) with the initial conditions from (2) tends to α as $t \to \infty$ in the case (a), tends to $-\alpha$ as $t \to \infty$ in the case (b) and tends to periodic solution in the case (c).*

If we replace $x(t)$ which is a solution of (1) with the initial conditions (2) with $-y(t)$, then it is obvious that $y(t)$ is also a solution with the initial conditions $\varphi(t) = -\xi(t)$ and $h = -\tilde{h}$ as a parameter. Thus, we showed that the case with decreasing initial conditions is reduced to the already considered.

RELAY MODEL OF A FADING NEURON

V. K. Zelenova

Centre of Integrable Systems, P.G. Demidov Yaroslavl State University, Yaroslavl, Russia; verzelenowa12@gmail.com

As a model of one neuron, the following equation is considered:

$$\dot{R} = \lambda\Big[\mathcal{F}\Big(R(t-h)\Big) + \mathcal{H}\Big(X_*(t)\Big)\Big]R(t), \tag{1}$$

proposed in the article [1]. Where $R(t)$ is a normalized membrane potential, $\lambda > 0$ is speed of electrical processes in a nerve cell, $\mathcal{F}$ is the threshold function characterizing the internal behavior of a neuron

$$\mathcal{F}(u) = \begin{cases} 1, & 0 < u \le 1, \\ -\alpha, & u > 1, \end{cases}$$

$$\mathcal{H}(u) = \begin{cases} -\eta, & 0 < u \le \theta, \\ \xi, & u > \theta, \end{cases}$$

$X_*(t) = e^{\lambda x_*(t)}$, $x_*(t)$ is the periodic function with period T_*

$$x_*(t) = \begin{cases} 0, & t = 0, \\ > 0, & 0 < t < t_*, \\ 0, & t = t_*, \\ < 0, & t_* < t < T_*, \end{cases}$$

$\eta, \xi, \alpha, \theta$ — positive parameters, $h > 0$ — time delay.

Equation (1) is a modification of the equation

$$\dot{u} = \lambda f(u(t-h))u, \tag{2}$$

proposed in the article [2]. Where $u = u(t) \ge 0$, $\lambda \gg 1$, function $f(x)$ is infinitely differentiable on $\mathbb{R}_+ = \{x \in \mathbb{R} : x \ge 0\}$ such that $f(0) = 1$, $f(x) \to -\alpha$ as $x \to +\infty$. This equation underlies a number of phenomenological neuromodels.

In the work [1] for the equation (1) the existence of solutions of a special form was proven: on some initial interval there is any predetermined number of equally high bursts, after which the solution immediately becomes periodic with small oscillations. Hereinafter, by the term "high bursts" we mean the values of the function $R(t)$ of the order of e^{λ}, by the term "small oscillations" of the order of $e^{-\lambda}$. Such a neuron is called "dying". The existence of solutions was also shown numerically, where after an initial interval with equally high bursts, the solution gradually transitions to small oscillations, the bursts decreasing their amplitude for some time. Such a neuron is called "aging".

In this work, we will expand these results in the sense that we will analytically show the existence and stability of the solution and find the range of parameters in which the solution moves from high periodic bursts to small oscillations immediately or gradually with a decrease in amplitude, which will combine the types of "aging" behavior and a "dying" neuron. We will call this generalization a "fading" neuron, where by the term "fading" we mean the transition to small oscillations.

This work was carried out within the framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher

Education of the Russian Federation (Agreement on provision of subsidy from the federal budget No. 075-02-2023-948).

REFERENCES

1. *Preobrazhenskaia, M. M.,* “Relay System of Differential Equations with Delay as a Perceptron Model”. In: Kryzhanovsky, B., Dunin-Barkowski, W., Redko, V., Tiumentsev, Y. (eds) Advances in Neural Computation, Machine Learning, and Cognitive Research VI. NEUROINFORMATICS 2022. Studies in Computational Intelligence, vol 1064. Springer, Cham (2023).
2. *Kolesov A. Y., Mishchenko E. F., Rozov N. K.,* “A modification of Hutchinson’s equation,” Comput. Math. and Math. Phys., 50, 1990–2002 (2010).

Научное издание

INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS
(ISND–2023)

ТЕЗИСЫ ДОКЛАДОВ КОНФЕРЕНЦИИ

4-я Международная конференция
"ИНТЕГРИРУЕМЫЕ СИСТЕМЫ И
НЕЛИНЕЙНАЯ ДИНАМИКА"
(ISND–2023)
Ярославль, 25—29 сентября 2023 г.

Компьютерный набор, верстка А.О. Толбей

Подписано в печать 24.10.23. Формат 60×90 1/16.
Усл. печ. л. 8,62. Заказ № 23141. Тираж 30 экз.

Отпечатано с готового оригинал-макета
в типографии ООО «Филигрань»,
150049, г. Ярославль, ул. Свободы, 91,
pechataet@bk.ru