P.G. Demidov Yaroslavl State University

Ярославский государственный университет им. П.Г. Демидова

6th International Conference on Integrable Systems & Nonlinear Dynamics

2025

INTEGRABLE SYSTEMS & NONLINEAR DYNAMICS (ISND–2025)

Yaroslavl, September 22−26, 2025

P. G. Demidov Yaroslavl State University
Regional Scientific and Educational Mathematical Center
“Centre of Integrable Systems”
“Tensor” IT company

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Abstracts

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RATIONAL INTEGRALS OF GEODESIC FLOWS ON 2-SURFACES

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We study 2-dimensional Riemannian metrics with integrable geodesic flows: an additional first integral is assumed to be rational in momenta. The existence of such integrable examples was proved in [1]. Later in [2] — [7] first examples of such metrics and rational integrals were constructed and investigated from various points of view.

In the talk we recall these known results and also describe one more integrable construction which allows to produce new explicit local rationally integrable examples of a very simple form.

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ON INTEGRABLE EXAMPLES OF MAGNETIC GEODESIC FLOWS ON THE 2-TORUS

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We study the Hamiltonian system

$$\dot{x}^j = \{x^j, H\}_{mg}, \qquad \dot{p}_j = \{p_j, H\}_{mg}, \qquad i, j = 1, 2 \tag{1}$$

on the 2-torus with the conformal coordinated x^1, x^2

$$ds^2 = \Lambda(x^1, x^2)(d(x^1)^2 + d(x^2)^2), \qquad H = \frac{p_1^2 + p_2^2}{2\Lambda(x^1, x^2)}$$

and the magnetic Poisson bracket

$$\{f, H\}_{mg} = \sum_{i=1}^{2} \left(\frac{\partial f}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial x^i}\right) + \Omega(x^1, x^2)\left(\frac{\partial f}{\partial p_1}\frac{\partial H}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial H}{\partial p_1}\right)$$

that admits an additional quadratic in momenta first integral F at a fixed energy level. System (1) defines the geodesic flow of the metric ds^2 in the magnetic field $\omega = \Omega(x^1, x^2)dx^1 \wedge dx^2$. In what follows, we denote $(x^1, x^2) = (x, y)$.

As proved in [1], condition $\{F, H\}_{mg} = 0$ at the fixed energy level $\{H = \frac{1}{2}\}$ can be written in the form of the quasi-linear system of PDEs

$$A(U)U_x + B(U)U_y = 0, \tag{2}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix},$$

and $U = (\Lambda, u_0, f, g)^T$. Coefficients of the first integral and the magnetic field are expressed in a certain way in terms of functions $\Lambda(x, y)$, $u_0(x, y)$, $f(x, y)$, $g(x, y)$.

System (2) admits various smooth and analytical solutions defined on the 2-torus ([2], [3]).

Also, as proved in [1], system (2) is semi-Hamiltonian, hence it is possible to apply the generalized hodograph method ([4]) to construct its solutions. To do this, we firstly describe symmetries of system (2) having the form

$$U_\tau = W(U)U_x. \tag{3}$$

Flows (2) and (3) must commute, therefore

$$\begin{gathered} VW - WV = 0, \\ (V_\tau - W_t + VW_x - WV_x)\, U_x = 0, \end{gathered} \tag{4}$$

where $V(U) = -B^{-1}A$.

Due to the generalized hodograph method, the following theorem holds true.

Theorem 1. *Suppose that matrices V and W satisfy satisfy (4). Then any smooth solution $\Lambda(x,y)$, $u_0(x,y)$, $f(x,y)$, $g(x,y)$ to the following system*

$$xE + tV = W$$

is also a solution to system (2).

In this work we, following this algorithm, construct new solutions to system (2) giving rise to integrable examples of magnetic geodesic flows with an additional quadratic integral at the fixed energy level. We note that firstly certain solutions to (2) obtained via the generalized hodograph method were described in [5].

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ON UNIQUENESS AND STABILITY OF THE LIMIT CYCLE OF MACKEY–GLASS EQUATION

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The work investigates the Mackey–Glass equation [1], which, after renormalization of the delay and an exponential substitution, takes the form

$$\dot{x}(t) = -\beta + \alpha \frac{e^{x(t-1)-x(t)}}{1+e^{\gamma x(t-1)}}, \tag{1}$$

where the parameters α, β, γ are positive.

Let σ_0, p, q be positive parameters satisfying the conditions

$$0 < p < \beta\sigma_0 < q, \quad \sigma_0 < \tfrac{1}{2}.$$

On the interval $[-1-\sigma_0, -\sigma_0]$, we define the set of initial functions

$$S = \{\varphi \in C[-1-\sigma_0, -\sigma_0] \mid 0 < p \leqslant \varphi(t) \leqslant q, \varphi(-\sigma_0) = \beta\sigma_0\}.$$

Considering γ to be a large parameter, we define the limiting object for (1) as $\gamma \to +\infty$.

$$\dot{x} = -\beta + \alpha e^{-x} F(\exp(x(t-1))), \tag{2}$$

$$\text{where } F(u) = \lim_{\gamma\to+\infty} \frac{u}{1+u^\gamma} = \begin{cases} 0, & u > 1, \\ \frac{1}{2}, & u = 1, \\ u, & 0 < u < 1. \end{cases}$$

Let the parameters $\alpha > \beta > 0$ satisfy the condition

$$\alpha > \exp\left(\beta(1+e^{-\beta})\right). \tag{3}$$

The following theorems are proved in [2].

Theorem 1. *Let condition (3) hold. Then, for any initial function $\varphi \in S$, equation (2) has a periodic solution $x^*(t)$ for $t \geqslant -\sigma_0$, described on the period T by the formulas*

$$x^*(t) = \begin{cases} -\beta t, & t \in [-\sigma_0, 1], \\ -\beta t + \ln(\alpha e^{\beta}(t-1)+1), & t \in [1,2], \\ -\beta t + \ln(\frac{\alpha^2}{2}e^{2\beta}(t-2)^2 + \alpha e^{\beta}(t-1)+1), & t \in [2, t_1], \\ -\beta t + \ln(\frac{\alpha^2}{2}e^{2\beta}(t_1-2)^2 + \alpha e^{\beta}(t_1-1)+1), & t \in [t_1, t_2]. \end{cases}$$

Here, t_1 is the minimal root of the equation

$$-\beta(t-1)+\ln(\alpha e^{\beta}(t-2)+1)=0,$$

$$T=\frac{1}{\beta}\ln\left(\frac{1}{2}\alpha^2e^{2\beta}(t_1-2)^2+\alpha e^{\beta}(t_1-1)+1\right),$$
$$t_2=1+T.$$

Theorem 2. *Let condition (3) hold. Fix a parameter $\nu\in(\frac{1}{2},1)$ and denote $\delta=\gamma^{-\nu}$. Equation (1) with an arbitrary initial function $\varphi\in S$ has a solution $x^*_\varphi(t,\gamma)$ with the asymptotics*

$$x^*_\varphi(t,\gamma)=x^*_\varphi(t)+O(\gamma^{1-3\nu}),$$

where the remainder term is uniform in $\varphi\in S$.

Theorem 3. *Let condition (3) hold. Then there exist values of the parameters σ_0, p, q and a sufficiently large γ_0 such that for all $\gamma>\gamma_0$, there exists an initial function $\varphi\in S$, depending on γ, for which the Mackey–Glass equation (1) has a periodic solution $x^*_\varphi(t,\gamma)$ with period T_γ. The following asymptotic equalities hold:*

$$\lim_{\gamma\to+\infty}\max_{t\in[-\sigma_0,T_\gamma-\sigma_0]}|x^*_\varphi(t,\gamma)-x^*_\varphi(t)|=0,\qquad \lim_{\gamma\to+\infty}T_\gamma=T.$$

The proof scheme for this theorem is as follows. We define the shift operator along the trajectory on the space of functions $C[-1-\sigma_0,-\sigma_0]$ with the supremum norm:

$$\Pi_\gamma(\varphi):t\mapsto x^*_\varphi(t+T_{\gamma,\varphi},\gamma),$$

where $T_{\gamma,\varphi}$ is defined as the second positive root of the equation

$$x^*_\varphi(t-\sigma_0,\gamma)=\beta\sigma_0,$$

which exists for sufficiently large γ.

With the correct choice of parameters p, q, σ_0, and for sufficiently large γ, the operator Π_γ maps the set of initial functions S into itself, whence, by Schauder's principle [3], the existence of a fixed point of the operator follows, which corresponds to a periodic solution of equation (1).

However, the uniqueness and stability of the limit cycle do not follow from these considerations. This work is devoted to proving the uniqueness and exponential orbital stability of the limit cycle $x^*_\varphi(t,\gamma)$ for sufficiently large γ.

The following theorem is proved in this work.

Theorem 4. The limit cycle $x^*(t,\gamma)$ of equation (1) is unique and exponentially orbitally stable.

The general proof scheme is analogous to [4]. It is proved that the Frèchet derivative of the operator Π_γ has the form

$$\partial_\varphi \Pi_\gamma(\varphi)[h] = y_h(t+T_{\gamma,\varphi}-\sigma_0) + \frac{y_h(T_{\gamma,\varphi}-\sigma_0)}{\dot{x}^*_\varphi(T_{\gamma,\varphi}-\sigma_0)}\dot{x}^*_\varphi(t+T_{\gamma,\varphi},\gamma),$$

where $h \in C[-1-\sigma_0,-\sigma_0]$, $h(-\sigma_0)=0$ is the variation of the initial function $\varphi \in S$, and the function $y_h(t)$ is determined from the equation

$$\dot{y}_h(t) = A(t)y_h(t) + B(t)y_h(t-1),$$

where

$$A(t) = -\alpha \cdot \frac{\exp(x^*_\varphi(t-1,\gamma)-x^*_\varphi(t,\gamma))}{1+\exp(\gamma x^*_\varphi(t-1,\gamma))},$$

$$B(t) = -\alpha \cdot \frac{\exp(x^*_\varphi(t-1,\gamma)-x^*_\varphi(t,\gamma)) \cdot \exp(\gamma x^*_\varphi(t-1,\gamma))(\gamma-1)}{(1+\exp(\gamma x^*_\varphi(t-1,\gamma)))^2},$$

$y_h(t) = h(t)$ for $t \in [-1-\sigma_0,-\sigma_0]$.

It is shown that the function $y_h(t)$ is exponentially small for $t \in [-\sigma_0, T_{\gamma,\varphi}-\sigma_0]$ uniformly on $\{h : \|h\|=1\}$, which leads to the estimate for the operator norm:

$$\sup_{\varphi\in S}\|\partial_\varphi \Pi_\gamma\| \leqslant M\exp(-A\gamma^{1-\nu}) \qquad (4)$$

for some positive constants M and A. This estimate, in turn, implies that Π_γ is a contraction, and hence the fixed point $\varphi^* \in S$, corresponding to the limit cycle $x^*(t,\gamma)$, is unique. Moreover, from (4) a similar estimate follows for the spectral radius of $\Pi_\gamma(\varphi^*)$. Since all multipliers μ of the cycle $x^*(t,\gamma)$, except the simple unit multiplier, are eigenvalues of this operator, they lie in the disk

$$\{\mu \in \mathbb{C} : |\mu| \leqslant M\exp(-A\gamma^{1-\nu})\},$$

which implies the exponential orbital stability of the limit cycle.

This work was carried out within the framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl

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NOTES ON A QUALITATIVE STUDY OF A SERIES OF DYNAMICAL SYSTEMS WITH POLYNOMIAL RIGHT-HAND SIDES IN A POINCARE DISK

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An extended series of dynamical systems having polynomial forms in their right-hand parts has been studied on the real plain of phase variables x, y. They have a common form:

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \tag{1}$$

where $X(x, y)$, $Y(x, y)$ are considered to be the mutually reciprocal polynomials, such as X be a cubic form, and Y be a quadratic form, while $X(0, 1) > 0$, $Y(0, 1) > 0$.

The problem of defining and constructing of all realizable (topologically different) phase portraits for the series of dynamical systems

(1) in the Poincare disk has been posed and solved. Close to coefficient criteria for the occurrence of each type of a phase portrait have been found. To solve this problem, the method of sequential mappings proposed by Henri Poincare was applied: 1) the central mapping of the plain x, y, together with the infinitely distant straight line $\bar{R}^2_{x,y}$ from the center $(0,0,1)$ of the sphere $\Sigma : x^2 + y^2 + z^2 = 1$ onto the Poincare sphere Σ, having its diametrically opposite points identified, 2) the orthogonal mapping of the lower enclosed hemisphere Σ onto the Poincare disk $\bar{\Omega} : x^2 + y^2 \leq 1$, having its diametrically opposite boundary points also identified [1, 2].

When studying this series of dynamical systems, a number of qualitative investigation methods were applied, developed specifically for the purposes of this research primarily by prof. A.F. Andreev and by the authors as well [3, 4, 7]. The initial extensive family of systems has been divided into subfamilies, which organize a row of levels of hierarchy, and their patterns of phase trajectories were classified according to this fact. As a result of this approach, the research algorithm includes several steps or sections.

On the first hierarchical level exist 10 different subfamilies of systems (1). This number appears to be the result of existence of 10 possible sequences of roots of the special characteristic polynomials, attached to the right-hand parts of equations (1). The example of this initial level gives the subfamily of systems called the (3,2)-systems, whose right-hand parts show 3 different multipliers of lower degree for the first equation and 2 for the second one:

$$X(x,y) = p_3(y-u_1x)(y-u_2x)(y-u_3x), \quad Y(x,y) = c(y-q_1x)(y-q_2x),$$

where $p_3 > 0$, $c > 0$, $u_1 < u_2 < u_3$, $q_1 < q_2$, $u_i \neq q_j$ for arbitrary i and j.

We name these systems the (3,2)-systems and introduce notations, common for all other types of subfamilies, i.e.
$P(u)$, $Q(u)$ — be the special polynomials of a system P, Q :

$$P(u) := X(1,u) \equiv p_3(u-u_1)(u-u_2)(u-u_3),$$
$$Q(u) := Y(1,u) \equiv c(u-q_1)(u-q_2);$$

RSP (RSQ) — be the ascending sequence of values of all the existing arithmetical roots of the systems' special polynomial $P(u)$ ($Q(u)$), RSPQ — be the ascending sequence of all arithmetical roots of the

both special polynomials $P(u)$, $Q(u)$.
The DC-transformation — be the double change: $(t, y) \to (-t, -y)$, which will transform the system into another one, having the opposite direction of movement along its phase trajectories, and also opposite signs of roots of polynomials $P(u)$, $Q(u)$, as well as the inverse numbering of those roots. If one dynamic system can be obtained via the DC-transformation of some other system, we call them the mutually inverse systems. Otherwise we call the two systems the independent ones regarding the DC-transformation. For the systems $(3, 2)$ have been found also 10 sub subfamilies, and 4 of them have a mutually inverse system. All topologically different phase portraits were revealed and constructed for all sub subfamilies, 93 types of portraits in a total.

Further at the first hierarchical level phase portraits were constructed for the 2 different classes of systems belong to (2,2)-subfamily and others, using the common algorithm [5, 6, 8].

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ON IMPLEMENTATION OF NUMERICAL METHODS FOR DISCONTINUOUS DIFFERENTIAL EQUATIONS

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We consider delay differential equations (DDEs) of the form

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_p), \dot{x}(t-\sigma_1), \ldots, \dot{x}(t-\sigma_q)\big), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the unknown function, and the delays τ_i, σ_j are positive and may be constant or dependent on time, the state, or delayed states. This general form encompasses ordinary differential equations (ODEs), retarded DDEs, and neutral DDEs as particular cases.

In this work, we focus on the case where the function f is discontinuous. More specifically, we consider systems in which f is piecewise continuous, with discontinuities occurring across smooth hypersurfaces. Such models frequently arise in the description of physical systems exhibiting abrupt changes in derivative values, where these rapid transitions are treated as instantaneous. Additionally, discontinuous differential equations naturally emerge in optimal control problems, where the control input often displays switching behavior.

In classical numerical analysis, the accuracy of methods such as Runge–Kutta schemes relies heavily on the smoothness of the solution. When the smoothness assumption is violated — e.g., when the solution exhibits discontinuities in its derivatives — the convergence order of higher-order methods deteriorates. For instance, applying Runge–Kutta methods to problems with discontinuities without proper handling may reduce the observed convergence order to that of the first-order Euler method [1]. Furthermore, even continuous DDEs can give rise to discontinuities if there is a mismatch between the derivative of the initial history function and the derivative dictated by the equation itself [2].

To address the challenge of discontinuities, we examine several strategies for handling transversal discontinuities during numerical integration:

- Ignoring discontinuities.
- Employing adaptive step size control.
- Detecting discontinuities via interpolation in continuous Runge–Kutta methods, with and without conservative switching during step calculation.
- Identifying discontinuities through stepwise iteration.

We develop and compare several Runge–Kutta-based numerical methods tailored for discontinuous ODEs and DDEs. The proposed implementations are evaluated with respect to their convergence order, stability properties, and computational performance. Additionally, we discuss the practical implications of incorporating these methods into general-purpose ODE/DDE solver libraries.

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BURSTING AND CHAOS OF THE REVIVING NEURON

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This work is devoted to the phenomenological modeling of neural cell behavior. We focus on a particular phenomenological neuron model represented by a delay differential equation [1,2]:

$$\dot{u} = \lambda \mathcal{F}_\alpha(u(t-1))u. \tag{1}$$

Here $u = u(t) > 0$ denotes the normalized membrane potential, the parameter $\lambda > 0$ characterizes the rate of electrical processes, and

$$\mathcal{F}_\alpha(u) := \begin{cases} -\alpha, & u \in (0,1], \\ 1, & u > 1, \end{cases} \qquad \alpha > 0. \tag{2}$$

The aim of this study is to investigate new effects that may arise in the association of neurons of type (1). For simplicity, we consider a pair of neurons: the first is isolated and described by equation (1), while the second is coupled to the first:

$$\begin{cases} \dot{u} = \lambda \mathcal{F}_\alpha(u(t-1))u, \\ \dot{v} = \lambda\big(\mathcal{F}_\beta(v(t-h)) + \mathcal{G}(u)\big)v, \end{cases}$$

where $u = u(t) > 0$ and $v = v(t) > 0$ denote the normalized membrane potentials of the neurons, λ characterizes the rate of electrical processes, $h > 1$ is the time delay, the functions $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ are given by formula (2), $\alpha, \beta > 0$, and

$$\mathcal{G}(v) := \begin{cases} -\eta, & v \in (0,1], \\ \xi, & v > 1, \end{cases} \qquad \xi, \eta > 0. \tag{3}$$

Thus, the first neuron acts as the driving one, while the second is driven. The equation describing the second neuron exhibits significantly richer dynamics compared to the first. In [3,4], the existence of a stable regime of a *fading neuron* was analytically proven. This regime is defined as follows. For any given fixed natural number n, one can choose parameters α, β, η, ξ, and h such that the function v is periodic with a large spike within the period on some initial time interval. Then, after a transient process during which the spike amplitude decreases, the function becomes periodic with small oscillations.

Another regime of the system is the so-called *reviving neuron*. Its essence is as follows. For any given fixed natural number n, one can choose parameters α, β, η, ξ, and h such that the function v exhibits n large spikes with increasing amplitude over some interval, after which the spikes gradually decay to small amplitude. Then, after some time, the solution pattern repeats. For the second equation of the system, the existence of periodic solutions was analytically established, and numerical simulations demonstrated the existence of chaotic regimes of the *reviving neuron* type. The first equation exerts a periodic influence on the second, thereby realizing a mechanism of chaos generation analogous to that arising under periodic forcing of a pendulum [5].

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AN ANALYZABLE PIECEWISE SMOOTH RECONSTRUCTION OF "BUTTERFLY FAMILY" CHAOTIC ATTRACTORS

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Despite the huge number of numerical and model studies of chaotic dynamical systems, rigorous statements about certain systems with given right-hand sides remain a rare exception.

In this talk, we propose piecewise smooth (PWS) three-dimensional ODE systems that qualitatively reproduce the main properties of Lorenz, Chen, Shimizu-Morioka, Lyubimov-Zaks and other systems with 3 eqillibria and "butterfly"-like attractors. In contrast to original smooth nonlinear systems, our PWS examples allow a rigorous study of chaotic attractors and their codimension-1 bifurcations. Our PWS models provide the global partition of the parameter space in contrast to all existing geometric models.

Our first PWS example reconstructs the classical dynamics of Lorenz-type systems with a homoclinic butterfly to a saddle with positive and negative saddle values. We show that in both cases the flow dynamics are completely and globally determined by the one-dimensional map [1]

$$\bar{x} = (1 - \gamma + \gamma |x|^{\nu}) \operatorname{sign} x$$

for $|x| \leq 1$, $0 \leq \gamma < \gamma_{cr} < 2$, $\nu > 0$. Here γ_{cr} is the value explicitly expressed in the parameters of the PWS system and corresponds to the appearance of sliding motions – a substantially non-smooth phenomenon, new in relation to the original smooth models [2].

For positive saddle value $\sigma \triangleq 1 - \nu > 0$, we prove the birth of a singular-hyperbolic attractor as a result of a heteroclinic bifurcation. For negative value $\sigma < 0$, we explicitly obtain homoclinic and pitchfork bifurcations forming a codimension-1 cascade leading to the emergence of a quasi-strange attractor [3].

The second PWS example reconstructs the dynamics of Lorenz-like systems in which the destruction of singular-hypebolic attractors occurs due to the appearance of non-orientable manifolds in their structure. We show that the derivation of the explicit Pouncaré map and the strong contraction condition lead to a one-dimensional map of the following form

$$\bar{x} = f(x) = (2 - |x|^{\nu})^{-\frac{\lambda}{\mu}} \cos\left[\frac{\omega}{\mu} \ln(2 - |x|^{\nu})\right] \operatorname{sign} x,$$

where $|x| < 1$, and ν, μ, λ, ω are positive parameters.

Using this map, we obtain a set of homoclinic "2- or 3-wings butterflies" that order bifurcation transitions from singular-hyperbolic attractors to quasi-strange attractors with non-orientable components.

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MODELING OF INSTABILITY CAUSED BY A SURFACE ELECTRIC CHARGE INDUCED ON A LIQUID SURFACE COVERED BY A SURFACTANT FILM

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The problem is considered about calculation of surfactant surface redistribution patterns following an instability of charged liquid surface. The instability arises undergo electric surface forces if they dominate on Laplace surface tension ones. As result the Taylor's cones form on the surface and form their tops a lot of small highly charged drops are emitted [1]. In frame of the model of ideal and perfectly conducive charged liquid of infinite depth with free surface the phenomenon is called Tonkes-Frenkel's Instability (TFI) [2]. The present work focuses on new features of the phenomenon specified by presence of an elastic surfactant film which covers horizontal charge liquid surface. The surface instability is considered relate to infinitesimal wave perturbations which also cause redistribution of local concentration of the surfactant film.

The mathematical formulation of the problem consists of equations of electro-hydrodynamics viscous incompressible perfectly conductive liquid. The standard boundary conditions are set on the horizontal liquid surface slightly distorted by the wave perturbation. The condition of continuity for surface concentration of the surfactant film is used to bundle the film property with the rest parameters of the problem. The film elasticity is supposed known constant. The problem is solved in linear approximation on a small parameter which one was relation of wave perturbation amplitude to wavelength. The dispersion equation and the time dependents for vertical deviation of surface and surfactant concentration were derived.

Analysis of the solution revealed that the critical surface charge density required initiating instability depends neither viscosity no elasticity. Its critical value directly defines on formula of the classical theory TFI [2]. If the surface charge density less than critical value the location of concentration surfactant maximum distinctly differ from ones of the wave top that fully accords to previously investigations [3]. But if the surface charge density exceeds critical value the concentration surfactant maximum locates right on the top of the increasing perturbation. That means thickening surfactant film on the top of Taylor's cone. So the drops emitted from Taylor's cone tends take surfactant portions away one by another.

The modeling of instability caused by surface electric charge induced on a liquid surface covered by a surfactant film discovered that the phenomenon assists cleaning of the surface from applied surfactant.

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POLYNOMIAL DYNAMICAL SYSTEMS ASSOCIATED WITH THE KDV HIERARCHY

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We will discuss g compatible polynomial dynamical systems with operators $\mathcal{D}_k$, where $k = 1, 3, \ldots, 2g-1$, in the space $\mathbb{C}^{3g}$ with coordinates $b_{i,j}$ for $i = 1, 2, 3$, $j = 1, 3, \ldots, 2g-1$ and $g \in \mathbb{N}$ given by the expressions

$$\mathcal{D}_1(b_{1,j}) = b_{2,j},\ \mathcal{D}_1(b_{2,j}) = b_{3,j},$$
$$\mathcal{D}_1(b_{3,j}) = 4(2b_{1,1}b_{2,j} + b_{2,1}b_{1,j} + b_{2,j+2}), \tag{1}$$

$$\mathcal{D}_k(b_{1,j}) = b_{2,j+k-1} + \sum_{s=1}^{(k-1)/2} b_{2,2s-1}b_{1,j+k-2s-1} -$$
$$- \sum_{s=1}^{(k-1)/2} b_{1,2s-1}b_{2,j+k-2s-1}, \tag{2}$$

$$\mathcal{D}_k(b_{2,j}) = \mathcal{D}_1(\mathcal{D}_k(b_{1,j})), \quad \mathcal{D}_k(b_{3,j}) = \mathcal{D}_1(\mathcal{D}_k(b_{2,j})), \tag{3}$$

where $b_{i,k} = 0$ for $k \geqslant 2g+1$.

We will show their relation to differentiation of genus g hyperelliptic functions as well as the Korteweg–de Vries hierarchy. The talk will present our recent development of this topic.

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CHAOTIC BEHAVIOR IN THE 2 BODY PROBLEM ON A SPHERE

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We prove the existence of chaotic trajectories for the two body problem on a sphere. The trajectories we construct encounter near-collisions and are similar to the second species solutions of Poincaré of the classical 3 body problem. The construction uses a general result on Lagrangian systems with Newtonian singularities of the potential which is based on the method of anti-integrable limit of Serge Aubry.

SINGULARITY ANALYSIS OF CHAOS IN NONLINEAR DYNAMICAL SYSTEMS

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In the early years of the 20th century, the French Mathematician Paul Painlevé and co-workers identified 50 ordinary differential equations (ODEs) of 2nd order as “solvable,” because their solutions have only poles, as movable (initial condition dependent) singularities and are hence single-valued in the complex t-plane (t being the independent variable). Some 40 years ago, this approach of “singularity analysis” was extended to differential equations which possess solutions that are infinitely sheeted in the time domain and chaotic in the sense of Mel’nikov. In this talk, I will review, in a pedagogical way, some work I had done in the 1980’s and 90’s in a direction inspired by V. V. Kozlov and co-workers, and which led to several interesting results of what I would call *singularity analysis of chaotic dynamical systems in complex time.*

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SOLUTIONS OF THE KADOMTSEV-PETVIASHVILI EQUATION IN HYPERELLIPTIC SIGMA FUNCTIONS AND BAKER FUNCTIONS

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Since the mid-90s, V. M. Buchstaber, V. Z. Enol'skii and D. V. Leikin have developed the theory of multidimensional sigma functions on Jacobians of algebraic curves in the direction of applications to problems of mathematical physics. In 1997, we gave an explicit construction of a basis for the field of meromorphic functions on hyperelliptic Jacobians and explicitly described the generators of the ideal of relations between these basis hyperelliptic functions. It is remarkable that these ideal generators, as differential equations, lead explicitly to the KdV equation. In this ideal, special cases of the KP-equation were also found.

The focus of the talk will be on the results of article T. Ayano, V. Buchstaber, arXiv:2502.19972v1 [math-ph] 27 Feb 2025. In this paper, explicit formulas for hyperelliptic solutions to general KP-equation was obtained and solved the long-standing problem of describing the dependence of these solutions on the variation of the coefficients of the hyperelliptic curve equation, which are integrals of the KP-equations.

THE GAUSSIAN MULTIPLICATIVE CHAOS FOR THE SINE-PROCESS

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Grigori Olshanski [1] posed the problem of deciding when two determinantal measures are mutually absolutely continuous and of finding the corresponding Radon—Nikodym derivative.

Olshanski proved that the determinantal point process on $\mathbb{Z}$ governed by the Gamma-kernel is quasi-invariant under the group of finite permutations of $\mathbb{Z}$ and computed the Radon—Nikodym derivative, a multiplicative functional given by a generalized Euler product.

In development of Olshanski's programme it has been shown ([2], Sept. 2014) that determinantal point processes governed by integrable kernels satisfying a regularity condition are quasi-invariant under the group of finite permutations in the discrete case and under the group of diffeomorphisms with compact support in the continuous case.

The Radon—Nikodym derivative is found explicitly as a regularized multiplicative functional over pairs of particles of our configuration. The key point is the equivalence of reduced Palm measures of the same order for our processes (that are rigid in the sense of Ghosh and Peres [8]): in this case the Radon—Nikodym derivative is given by a regularized multiplicative functional over particles of our process, in other words, a random Euler product. In the particular case of the sine-process the square root of the resulting random entire function coincides, up to a linear multiple, with the "stochastic zeta-function" subsequently studied by Chhaibi, Najnudel, Nikeghbali [6].

The random Euler product obtained as the square root of the Radon—Nikodym derivative of Palm measures of a determinantal point process naturally arises in the problem of minimality of realizations of our process in the Hilbert space that governs it.

In joint work with Yanqi Qiu and Alexander Shamov [3] it is proved that almost every realization of a determinantal point process governed by an orthogonal projection is a complete set for the underlying reproducing kernel Hilbert space. The result had been conjectured by

Lyons and Peres; in the case of the discrete space, it had been proved by Lyons; in the rigid case and, in particular, for the sine process, by Ghosh [7].

Our complete set is however not minimal: indeed, almost every realization of the sine process has excess 1 for the Paley—Wiener space, that is, the configuration stays complete and becomes minimal after one particle is removed [4]. The reason for the excess one is that the suitably rescaled random Euler product converges in distribution to the Gaussian multiplicative chaos, a random measure, introduced by Mandelbrot, Peyrière and Kahane in development of the 1941 theory of the local structure of turbulence by Andrei Nikolaevich Kolmogorov [9].

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A VARIATIONAL FORMULATION OF HITCHIN'S INTEGRABLE SYSTEM WITH LAGRANGIAN MULTIFORM THEORY

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We will briefly review the ideas of Lagrangian multiform theory. Then, we will explain how to use it to introduce a variational formulation of the Hitchin system on a compact Riemann surface with and without marked points. This is achieved by first introducing a gauged version of phase space Lagrangian 1-forms. The construction is then applied to the phase space of the Hitchin system. In the case with marked points, our results provide a multiform version of the so-called 3d mixed BF action with defects, a mixed holomorphic-topological gauge theory designed for finite dimensional integrable systems, in the same way as the so-called 4d Chern-Simons gauge theory was designed for two-dimensional integrable field theories. The explicit reduction of our 3d mixed BF action with defects by a moment map constraint yields a unifying action and Lagrangian 1-form whose Euler-Lagrange equations produce the Hitchin system in the form of a hierarchy of compatible Lax equations. A special case reproduces the Lagrangian 1-form for the rational Gaudin hierarchy obtained previously in the algebraic framework of the classical r-matrix. The case of genus 1 with marked points gives a Lagrangian 1-form for the elliptic Gaudin hierarchy which contains as a special case the elliptic spin Calogero-Moser hierarchy. Our results establish the gauge theoretic origin of these important classes of finite dimensional integrable hierarchies within the entirely variational framework of Lagrangian multiforms.

GEOMETRY OF PERIODIC FLOWS OF HETEROGENEOUS FLUIDS: ASYMPTOTIC THEORY AND LABORATORY EXPERIMENTS

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Under the action of gravity g (which defines the vertical axis z of a Cartesian frame), heterogeneous fluids — whose density $\rho(z)$ depends on pressure, temperature, dissolved substances, and suspended particles — are naturally stratified: lighter volumes rise while heavier ones sink. Stratification is characterized by a buoyancy length scale $\Lambda = |d \ln(z) / dz|^{-1}$ as well as the buoyancy frequency $N = \sqrt{g/\Lambda}$ or period $T_b = 2\pi/N$ correspondently. In laboratory settings, stratification is typically strong ($T_b \sim 10$ s), whereas in the natural environment it is weak ($T_b \sim 10$ min); in a potentially homogeneous fluid it is almost imperceptible ($T_b \sim 10$ days) [1]. Accounting for spatial density variations allows calculating both the dynamics and the geometry of flows and to compare the results with optical observations. Because the refractive index of many transparent media varies linearly with density, interpreting such observations is simplified. Approximation of actually homogeneous fluid ($\rho = const$, $N \equiv 0$) omit the ability to describe flow structures is lost.

Modern mechanics of heterogeneous media is based on a system of fundamental equations — the continuous analogues of the conservation laws for mass, momentum, substances, and total energy — supplemented by constitutive relations (equations of state) for the Gibbs potential, density, and adiabatic speed of sound [2, 3], along with physically justified initial and boundary conditions. The formulation takes into account the granular distribution of Gibbs potential produced by associates of diverse physical and chemical origin. As phase boundaries, the shells of these associates store surface potential energy. Modeling flows, either in full or reduced form, the main energy — transfer mechanisms are considered: radiative, translational (via currents, waves, and vortices), dissipative, and conversion processes associated with the decay and formation of associates as well as of the free surface of the fluid. The interruption of molecular fluxes at impermeable boundaries in a stratified medium produces the most common class of fluid motions — fine diffusion induced flows on topography [4].

A complete asymptotic analysis of linearized fundamental equations, carried out by singular-perturbation techniques under compatibility conditions, shows that periodically acting external factors generate a broad family of periodic flows in a rotating, compressible, heterogeneous medium. The studied family includes inertial waves, surface and internal gravity waves, capillary waves, acoustic waves, hybrid modes, and their accompanying ligament structures. Direct nonlinear interactions among all flow components lead to rapid evolution of the overall flow pattern [5]. Calculated internal waves and associated ligaments agree satisfactorily with schlieren and electrolytic visualization pattern of flows produced in a laboratory basin by an oscillating body as well as uniformly moving sphere or plate [4, 6]. Ligaments are also observed when a freely falling drop merges with a quiescent liquid over a wide range of parameters [7].

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ASYMPTOTIC ANALYSIS OF THE PROPAGATION OF A TWO-DIMENSIONAL FRONT THROUGH THE INTERFACE OF THE MEDIUM

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The problem for a two-dimensional reaction-diffusion equation with cubic nonlinearity is considered. The solution of propagating front type is investigated, and the peculiarity of the problem lies in the presence of a discontinuity on the right side of the equation.

The following problem is investigated

$$\begin{cases} \varepsilon^2\Delta u - \varepsilon\dfrac{\partial u}{\partial t} = \big(u-\varphi^{(-)}(x,y)\big)\big(u-q(x,y)\big)\big(u-\varphi^{(+)}(x,y)\big), \\ x\in\mathbb{R}, y\in(0,a), t\in(0,T], \\ u_y(x,0,t,\varepsilon)=u_y(x,a,t,\varepsilon)=0, \quad x\in\mathbb{R}, t\in[0,T], \\ u(x,y,t,\varepsilon)=u(x+L,y,t,\varepsilon), \quad x\in\mathbb{R}, y\in[0,a], t\in[0,T], \\ u(x,y,0,\varepsilon)=u_{init}(x,y), \quad x\in\mathbb{R}, y\in[0,a]. \end{cases}$$

$$q(x,y)=\begin{cases} q_l(x,y), \quad y\le h_0(x), \quad q_l(x,h_0(x))\ne q_r(x,h_0(x)), \\ q_r(x,y), \quad y>h_0(x), \end{cases}$$

where ε is a small parameter, $h_0(x)$ is a smooth curve along which there is a drastic change in the properties of the medium and $u_{init}(x,y)$ is a continuous function. Let us denote the right side of the equation as $f(u,x,y)$. Functions $f(u,x,y)$ and $u_{init}(x,y)$ are L-periodically by variable x ($L>0, L=const$).

A two-dimensional equation of the reaction-diffusion type is investigated in the case when an autowave front passes through the interface of the medium. In the problem, a non-stationary curve $h(x,t)$ describes the position of the front, while another smooth stationary curve $h_0(x)$ describes the discontinuity of the medium characteristics. The local Vishik-Lusternik coordinates are used to describe the transition layer and the front near the interface of the medium. Using the example, the position of the curves $h(x,t)$ and $h_0(x)$ is considered in the case when they are close enough and their small neighborhoods intersect. Also, an asymptotic approximation of the solution of moving front type is constructed.

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NUMERICAL-ASYMPTOTIC METHOD FOR DETERMINING THERMOPHYSICAL CHARACTERISTICS OF HIGH-TEMPERATURE FUEL ALLOYS AND COMPOSITES USING SIMULATION DATA

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One of the main problems of calculating the fuel elements of nuclear reactors (FE) is the thermal calculation - determination of the thermal state of the FE (total heating temperature, thermal gradients and flows). This information is necessary for the next stage of the FE calculation, associated with the strength analysis of the heat-removing elements.

For most FE configurations and types of nuclear fuels, calculations use the linear dependence of the thermal conductivity coefficient on temperature and the estimation of heat fluxes according to Fourier’s law [1]. A change in the thermal conductivity of the fuel cores with increasing temperature leads to high temperature gradients, which, in turn, contributes to a change in the structure of the material, compaction and other deformations of the fuel cores. These effects are taken into account when designing FE.

In order to ensure reliable and efficient operation of fuel elements, resistant to emergency situations, new types of nuclear fuel are created, for example, uranium and plutonium alloys and composites based on metal alloys [2]. The thermophysical characteristics of new fuel materials are determined by calculation [3] using analysis and generalizations [4] and are an important component of solving the problem of providing nuclear power generation with raw materials.

We have developed and strictly substantiated an effective mathematical method for determining thermal conductivity coefficients and estimating heat flows in nonlinear media using modeling data. The method is based on a stable numerical algorithm for solving the inverse problem of reconstructing unknown parameters of a singularly perturbed model for the nonlinear heat conductivity equation (in this case, with a flux term linearly dependent on temperature), which uses an asymptotic approximation by a small parameter of the solution of the direct problem [5] in combination with experimental data based on known results and methods [6]. Numerical calculations were performed using the parameters of real fuel kernels for some innovative types of fuel compositions based on uranium, plutonium and zirconium, as well as nitride nuclear fuel.

The study was carried out within the framework of the state assignment of Lomonosov Moscow State University.

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ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS WITH RESPECT TO LARGE NUMBERS AND SOME QUESTIONS ABOUT THE INTEGRABLE HAMILTONIAN SYSTEMS THEY GENERATE

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We discuss the approach of constructing uniform asymptotic formulas in the form of special functions of a complex argument of various orthogonal polynomials with one or more indices. The asymptotics are constructed for large numbers. We discuss approaches based on both recurrence equations for polynomials and differential equations that these polynomials satisfy. One of the main examples is the joint orthogonal Hermite polynomials with two indices. The difficulties in applying the standard quasiclassical approach are related to the complexity of the symbols (which generate classical Hamiltonian systems) of the corresponding operators. We show that for this type of problem, the Feynman-Maslov operator calculus allows us to reduce the original problems to a set of problems with real-valued symbols and use the canonical Maslov operator to construct their asymptotic solutions. The proposed constructions give rise to a number of problems in the theory of integrable systems for Hamiltonians with special properties. The special properties of such systems and further transformations of the obtained formulas allow us to obtain uniform asymptotic formulas in the form of special functions that generalize and simplify the known Plancherel-Rothach formulas in the theory of 1-D Hermite polynomials.

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MULTIDIMENSIONAL RIEMANNIAN METRICS IN THEORY OF THE NAVIER-STOKES EQUATIONS

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To integrating of Navier-Stokes equations on base of $14D$-the Ricci-flat $R_{ik} = 0$ metrics we use the $6D$ and $4D$- the Ricci-flat metrics, associated with the KP- and KdV-equations defined by the system of linear equations of the form $\frac{\partial}{\partial y}\psi(x,y,z) + \frac{\partial^2}{\partial x^2}\psi(x,y,z) - 4/3\,\frac{\partial}{\partial x}h(x,y,z) = 0$, $\frac{\partial}{\partial z}\psi(x,y,z) - \frac{\partial}{\partial y}h(x,y,z) - \frac{\partial^3}{\partial x^3}\psi(x,y,z)$ $-3/2\,\psi(x,y,z)\,\frac{\partial}{\partial x}\psi(x,y,z) + \frac{\partial^2}{\partial x^2}h(x,y,z) = 0$, and which is used then to construct solutions of the KP and KdV -equation.

$$6\,\psi(x,y,z)\,\frac{\partial^2}{\partial x^2}\psi(x,y,z) + 6\,\left(\frac{\partial}{\partial x}\psi(x,y,z)\right)^2 + 3\,\frac{\partial^2}{\partial y^2}\psi(x,y,z) -$$

$$4\,\frac{\partial^2}{\partial z\partial x}\psi(x,y,z) + \frac{\partial^4}{\partial x^4}\psi(x,y,z) = 0.$$

$$\frac{\partial}{\partial z}\psi(x,y,z) - \frac{\partial}{\partial y}h(x,y,z) - \frac{\partial^3}{\partial x^3}\psi(x,y,z) - 3/2\,\psi(x,y,z)\,\frac{\partial}{\partial x}\psi(x,y,z) +$$

$$\frac{\partial^2}{\partial x^2}h(x,y,z) = 0.$$

For the KP-equation we get

$$6\psi(x,y,z)\frac{\partial^2}{\partial x^2}\psi(x,y,z) + 6(\frac{\partial}{\partial x}\psi(x,y,z))^2 + 3\frac{\partial^2}{\partial y^2}\psi(x,y,z) -$$
$$4\frac{\partial^2}{\partial z\partial x}\psi(x,y,z) + \frac{\partial^4}{\partial x^4}\psi(x,y,z) = 0.$$

The Ricci-flat $6D$ metric in the local coordinates (x,y,t,u,v,w) has applications in theory of moduli of Riemann surfaces and corresponding them the Theta-functions of algebraic curves.

To integrating Navier-Stokes equations with help of $14D$ flat metric $R_{ik} = 0$ we use the Ricci flat metrics $6D$ and $4D$-spaces associated with the KP and KdV-equations, defined by the system of linear equations of the form $\frac{\partial}{\partial y}\psi\,(x,y,z)+\frac{\partial^2}{\partial x^2}\psi\,(x,y,z)-4/3\,\frac{\partial}{\partial x}h\,(x,y,z)=0$, $\frac{\partial}{\partial z}\psi\,(x,y,z)-\frac{\partial}{\partial y}h\,(x,y,z)-\frac{\partial^3}{\partial x^3}\psi\,(x,y,z)-3/2\,\psi\,(x,y,z)\,\frac{\partial}{\partial x}\psi\,(x,y,z)+\frac{\partial^2}{\partial x^2}h\,(x,y,z)=0$.

The Ricci-flat $6D$ metric in the local coordinates (x,y,t,u,v,w) in the theory of moduli of Riemann surfaces and corresponding the Theta-functions of algebraic curves is used. It is in form $ds^2 = 4\,u(\frac{\partial}{\partial x}P(x,y,t))d_xd_t + 2\,d_xd_u + 4u(\frac{\partial}{\partial y}P(x,y,t))d_yd_t + 2\,d_yd_v + (-2uP(x,y,t)\frac{\partial}{\partial x}P(x,y,t) - 2u\frac{\partial^3}{\partial x^3}P(x,y,t) - 2\mu v\frac{\partial}{\partial y}P(x,y,t) + 2\,w\frac{\partial}{\partial x}P(x,y,t))d_t{}^2 + 2d_tdw$. The above property of the linear system is used to construct solutions of the KP equation by the introducing an auxiliary parameter τ. The auxiliary parameter for constructing particular solutions of equations in the partial differential equation was considered earlier in the author's works and it is fairly effective procedure.

TEMPERATURE DEPENDENT ELASTIC AND STRUCTURAL PROPERTIES OF UNIAXIALLY STRAINED GRAPHENE

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Graphene is known for its exceptional material properties, both mechanical and electronic [1,2]. We study the mechanical and structural properties of graphene under the effects of temperature and uniaxial strain, using the formalism of Hamiltonian dynamics, and a symplectic integrator to model the in-plane molecular dynamics of this material in time [3,4]. This method allows fast numerical integration while the numerical error is upper-bounded by a constant for all time [5]. In particular, we quantify the stress-strain response of graphene at various temperatures, allowing comparison to other investigations which use different modelling techniques [6,7]. We also study the effects of strain and temperature on the distributions of bond lengths and bond angles in the lattice [8], and provide analytic expressions for these distributions using a combination of analytical and numerical observations of the material behaviour.

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CUBIC-QUADRATIC NONLINEARITY EFFECTS ON STABLE REGIMES IN DYNAMICALLY SELF-ORGANIZING SYSTEMS

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Consider the system of differential equations:

$$\frac{du_n}{dt} = -N(v_{n+1} - v_n) + \mu u_n - u_n^3, \quad \frac{dv_n}{dt} = -N(u_n - u_{n-1}), \tag{1}$$

$$u_0 = 0, \quad v_{N+1} = \beta \frac{du_N}{dt}, \quad n = 1, 2, \ldots, N, \tag{2}$$

modeling an association of nonlinear oscillators. Of particular interest is the regime when $N \gg 1$, prompting the transition to a continuous model:

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x} + \mu u - u^3, \quad \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x}, \tag{3}$$

$$u|_{x=0} = 0, \quad \left(\beta \frac{\partial v}{\partial x} + v\right)\Big|_{x=1} = \beta(\mu u - u^3)\big|_{x=1}. \tag{4}$$

These models were studied for small μ analytically and numerically in [1].

We consider a modified system (1)-(2) with quadratic nonlinearity in the right-hand side:

$$\frac{du_n}{dt} = -N(v_{n+1} - v_n) + \mu u_n - \gamma u_n^2 - u_n^3, \quad \frac{dv_n}{dt} = -N(u_n - u_{n-1}), \tag{5}$$

$$u_0 = 0, \quad v_{N+1} = \beta \frac{du_N}{dt}, \quad n = 1, 2, \ldots, N. \tag{6}$$

and the corresponding continuous model:

$$\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x} + \mu u - \gamma u^2 - u^3, \quad \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x}, \tag{7}$$

$$u|_{x=0} = 0, \quad \left(\beta \frac{\partial v}{\partial x} + v\right)\Big|_{x=1} = \beta(\mu u - \gamma u^2 - u^3)\big|_{x=1}. \tag{8}$$

We distinguish two cases: $\gamma = C\mu$, $\mu \ll 1$ and γ of order 1.

For the regime $\gamma = C\mu$ with $\mu \ll 1$, asymptotic analysis of the complex amplitude evolution equations governing the critical modes of boundary value problem (7)-(8) yields a quasi-normal form (9):

$$\dot{\xi}_l = \left[\delta_l - d_l\,|\xi_l|^2 - \sum_{\substack{m=1 \\ m\neq l}}^{\infty} d_{l,m}\,|\xi_m|^2\right]\xi_l, \quad l \geq 1, \tag{9}$$

identical to that derived for system (3)-(4). This asymptotic reduction, performed via standard multiple-scale perturbation techniques (cf. [1-3]), establishes that boundary value problem (7)-(8) possesses, analogously to problem (3)-(4), a family of single-mode equilibrium states parametrized by the natural number l:

$$O_l = \{(\eta_1, \eta_2, \ldots, \eta_m, \ldots) : \ \eta_l > 0, \quad \eta_m = 0 \ \text{for all} \ m \neq l\}.$$

Numerical simulations were performed to examine the dynamical evolution of both systems (3)-(4) and (7)-(8) under identical initial conditions derived from the quasi-normal form analysis. The comparative study tracks solution trajectories as the parameter μ varies from the asymptotic regime to $O(1)$ values, with the proportionality constant fixed at $C = 1$.

For the regime where $\gamma = O(1)$, asymptotic analysis reveals a fundamental departure from the previous case: while the quasi-normal forms governing systems (3)-(4) and (7)-(8) possess identical functional structures (9), they exhibit distinct coefficient values, leading to qualitatively different dynamical behavior. Extensive numerical investigations of system (7)-(8) dynamics were conducted across a two-parameter domain, systematically varying μ from the perturbative regime ($\mu \ll 1$) to finite amplitude values ($\mu = O(1)$), while simultaneously increasing γ from the weakly nonlinear regime ($\gamma = O(\mu)$) to the strongly nonlinear regime ($\gamma = O(10)$). These computational studies demonstrate the bifurcation and stabilization of coherent spatiotemporal patterns exhibiting significantly enhanced morphological complexity compared to the single-parameter case.

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EXTREME OCEAN EVENTS ARISING FROM IRREGULAR WAVES

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In this talk, we begin by examining the emergence of extreme ocean waves – commonly referred to as rogue or freak waves – which present serious hazards to coastal regions, small vessels, and swimmers. Remarkably, analogous wave behaviors have been reported in other disciplines, including plasma physics and financial systems. The formation of such waves is investigated through Monte Carlo simulations, either by modeling ensembles of solitons with randomly distributed amplitudes – known as soliton turbulence – or by representing the wave field as a superposition of harmonics with random phases. We further analyze key statistical properties of these fields, including their spectra, higher-order moments, and probability distribution functions.

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DYNAMIC SELF-ORGANIZATION IN NETWORKS OF NONLINEAR OSCILLATORS

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The concept of dynamic self-organization is introduced. Let's assume a set of free (non-interacting) neurons, each of which is quiescent or incapable of oscillatory electrical activity. Then, when connected in a specific network, these neurons can begin to generate electrical impulses. The feasibility of this phenomenon is illustrated using a mathematical model representing a nonlinear hyperbolic boundary value problem. Using a combination of analytical and numerical methods, the question of attractors is investigated.

The phenomenon of self-organization is a fundamental concept in various fields of natural science, including physics, biology, neurodynamics, and others. In the most general terms, the essence of this phenomenon lies in the ability of a system to spontaneously (without any special external influences) transition from a simple state to a more complex one. In the particular case of a system consisting of multiple so-called partial or elementary subsystems, the principle of self-organization can be interpreted as follows: it may consist of the emergence of certain new properties in interacting subsystems that are clearly absent from each individual partial system.

In the case of neural networks, a partial system is naturally considered to be an individual neuron. The basic property of such a subsystem is the neuron's ability to generate electrical oscillations. This paper demonstrates that a situation is possible where, after being combined into a network, previously "silent" neurons, or even those incapable of generating impulses at all, begin to exhibit electrical activity. This phenomenon can appropriately be called dynamic self-organization.

The goal of this paper is to propose and explore a mathematical model of the phenomenon of dynamic self-organization. The corresponding problem is, in turn, divided into two tasks. First, it is necessary to select a mathematical model of an individual neuron within which stable self-oscillatory modes are absent. Secondly, it is necessary to find a suitable method of coupling partial systems such that the corresponding network, conversely, would allow stable self-oscillations.

The phenomenon of dynamic self-organization manifests itself in the increasing complexity of solutions to the systems under study in the presence of connections between oscillators. We considered two cases: in one, the corresponding dynamic system, with suitable parameter values, possesses an increasing number of coexisting simple attractors (cycles or tori), while in the other, chaotic oscillatory regimes of increasing Lyapunov dimension emerge. Both are manifestations of the behavioral complexity of the corresponding association (see [1–3]).

It can be assumed that the phenomenon of dynamic self-organization is related to the problem of the complex behavior of neural associations. The results presented in this paper open the possibility of substantiating a new approach to the problem of the complex behavior of neural networks. To clarify the essence of the matter, we will call the connections emerging in a neural association creative if they lead to the emergence of the phenomenon of dynamic self-organization (the existence of such connections has been demonstrated theoretically by us using the mathematical models discussed above). Our approach to the question of the nature of the complex behavior of neural associations presupposes the emergence of creative connections.

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HOMOCLINIC RESONANCES IN AREA-PRESERVING MAPS, POINCARE PROBLEMS AND STRUCTURES LIKE "SATURN-RING"

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We consider analytic two-dimensional area-preserving maps that have quadratic homoclinic tangency of invariant manifolds of a saddle fixed point. For such maps, we specify homoclinic resonance conditions under which the map has, in a small neighborhood of the orbit of homoclinic tangency, a countable set of generic (KAM-stable) elliptic points of all successive periods $k_0, k_0 + 1, ...,$ beginning with some k_0.

The set of such orbits forms a specific cluster of an (infinitely) large number of stable points. This cluster appears under a codimension 2 global bifurcation and, hence, it is not structurally stable as a whole even with respect to conservative perturbations. However, under the property "any fixed number of points", it becomes stable. And then we can talk about the possibility of observing this phenomenon. The question of where this can be observed, if we consider it in retrospect, can be directly related to those problems of celestial mechanics that H.Poincaré considered in his famous "New Methods of Celestial Mechanics", and in particular, to his planar circular restricted problem of three bodies. In this problem, each elliptic periodic orbit can be compared with some cosmic body, not necessarily large, it can be a stone, a piece of ice, etc. Individually, such a body can be "invisible", but when many of them gather in one place, then this can already be an observable cosmic object. Objects of this type are well known, the most famous being the ring of Saturn. It seems clear that some global resonance must be the cause of such a phenomenon. What its nature is, if so many small bodies are involved in this resonance, is not very clear. But in light of this work, we can assume that this is something like the homoclinic resonance we have considered.

Conclusions: 1) Mathematical.

(A) We have found effectively verifiable conditions under which two-dimensional area-preserving maps have infinitely many generic elliptic periodic points.

(B) We have shown that such points form a certain cluster in which all its points are ordered by their orbits with periods k, k+1, :

(C) All points are concentrated in one region: they run for a period along a ring-like neighborhood of the homoclinic orbit.

2) Physical (celestial mechanics). It seems that, following the approach of Poincaré who combined Hamiltonian dynamics and celestial mechanics, we can propose some mechanism for the emergence of global cosmic resonances involving a large number of small bodies in one cluster (like the ring of Saturn).

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ALGORITHM FOR CONSTRUCTING ASYMPTOTICS OF PERIODIC SOLUTIONS IN LASER MODELS WITH A RAPIDLY OSCILLATING DELAY

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The problem of stability of the equilibrium state in a laser system with fast oscillating coefficients is considered. A system averaged over fast oscillations and with a distributed delay is constructed. Critical cases in the problem of the stability of the equilibrium state are singled out. It is shown that the threshold value of the feedback coefficient at which the equilibrium state becomes unstable increases due to rapid oscillations compared to the corresponding value in the absence of modulation. In critical cases, normal forms are constructed – equations for the slowly varying amplitude of the periodic solutions. The conditions for the existence, stability and instability of cycles are revealed.

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ON MATRIX LAX PAIRS UNDER THE GAUGE EQUIVALENCE RELATION AND MIURA-TYPE TRANSFORMATIONS FOR LATTICE EQUATIONS

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In this talk, which is based on the papers [1, 2], I will present new results on interconnections of differential-difference matrix Lax representations (Lax pairs), gauge transformations, and discrete Miura-type transformations (MTs), which belong to the main tools in the theory of integrable differential-difference (lattice) equations.

Such equations are currently the subject of intensive study, since they occupy a prominent place in the modern theory of integrable systems and its applications in mathematical physics and geometry. In particular, such equations emerge in discretizations of integrable partial differential equations (PDEs), in discretizations of various differential geometric constructions and as chains associated with Bäcklund and Darboux transformations of PDEs (see, e.g., [3, 4, 5, 6] and references therein).

For a given equation, two matrix Lax representations (MLRs) are said to be gauge equivalent if one of them can be obtained from the other by means of a (local) matrix gauge transformation. Matrix gauge transformations constitute an infinite-dimensional group called the matrix gauge group, which acts naturally on the set of MLRs of a given equation. Two MLRs are gauge equivalent if and only if they belong to the same orbit of the matrix gauge group action.

For MLRs of (vector) evolutionary differential-difference equations, I will present results on the following questions:

1. When and how can one simplify a given MLR by matrix gauge transformations and bring the MLR to a form suitable for constructing MTs?

2. A MLR is called fake if it is gauge equivalent to a trivial MLR. How to determine whether a given MLR is not fake?

As an application, this allows one to construct new integrable equations (with new MLRs) connected by new MTs to known equations.

The considered examples include the following:

- a 3-component lattice introduced by D. Zhang and D. Chen in their work on Hamiltonian structures of evolutionary lattice equations [7],
- some rational 1-component equations of order $(-2, 2)$ related to the Narita–Itoh–Bogoyavlensky lattice,
- the 2-component Boussinesq lattice related to the lattice W_3-algebra,
- a number of 2-component equations connected to the Toda lattice written in the Flaschka–Manakov coordinates,
- a 2-component equation (introduced by G. Marí Beffa and Jing Ping Wang in their work on Hamiltonian evolutions of polygons [5]) which describes the evolution induced on invariants by an invariant evolution of planar polygons.

Also, several new integrable differential-difference equations connected by new MTs to the above-mentioned examples will be presented in the talk.

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ASYMPTOTIC INTEGRABILITY AND ITS CONSEQUENCES

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The notion of *asymptotic integrability* is based on the asymptotic theory of propagation of high-frequency wave packets along large-scale and time-dependent backgrounds. We assume that the evolution of the background obeys the dispersionless (hydrodynamic) limit of the nonlinear wave equation under consideration and demand that the Hamilton equations for the packet's propagation have an additional integral of motion independently of the initial conditions for the background dynamics. This condition is studied for systems described by two wave variables, and it is shown that it imposes strong restrictions on the dispersion relation for linear harmonic waves. If the condition of asymptotic integrability is fulfilled, then the additional integral and the dispersion relation define the quasiclassical limit of Lax pairs for completely integrable equations [1]. We show for several typical examples that the full expressions for the Lax pairs can be restored after a proper "quantization" of the quasiclassical relationships [2].

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NEW LORENZ-TYPE ATTRACTORS OF THREE-DIMENSIONAL ODES AND MAPS

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The foundations of the theory of hyperbolic attractors were laid more than sixty years ago in the classical works of Anosov, Smale, Sinai, Williams, Plykin, Pesin, Newhouse, Ruelle and others. The dynamics of such attractors is not sensitive to small variations, moreover, such attractors are structurally stable or rough. The first examples of such attractors in physical systems were discovered and studied by S.P. Kuznetsov [1]. It should be noted that only a few physical systems demonstrating hyperbolic chaotic attractors are known so far. At the same time, the search for new such systems is considered futile due to strong restrictions that ensure hyperbolicity. The most important relaxation of the conditions of hyperbolicity and singular hyperbolicity of attractors was the creation of the concept of pseudohyperbolic attractors. The theory of such attractors was constructed by D. Turaev and L. P. Shilnikov [2, 3].

The report is devoted to the construction and study of new examples of robust (pseudohyperbolic) attractors, including those included in applied physical models. To date, only a few examples of pseudohyperbolic attractors are known. Moreover, the overwhelming majority of them are abstract mathematical objects that are not encountered in applications. As part of the work, examples of new types of pseudohyperbolic attractors in three-dimensional systems of ordinary differential equations (as well as their discrete analogs) were discovered for the first time - in particular, conjoint Lorenz attractors (in system of ODEs with axial and central symmetry which is a normal form of bifurcation of equilibrium state with triply zero eigenvalues), as well as two-winged and four-winged Simo attractors (in system of ODEs with Z_4 symmetry which is a normal form of bifurcation of fixed point with $(-1, i, -i)$-multipliers). The corresponding geometric models were constructed and criteria for the existence of such attractors were proposed. A significant result is the demonstration that Simo attractors can thus arise in applied models, in a series of coupled oscillators, which confirms the physical feasibility of the constant mathematical model.

The developed methods and criteria for detecting robust attractors open up opportunities for a targeted search for this type of attractors in a wide class of dynamic systems, which can provide a deeper understanding and control over complex dynamics in applied problems.

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NONLOCAL CYCLES IN NONLINEAR SYSTEM WITH DELAY AND SMALL PARAMETER

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We consider a system of differential equations

$$\dot{u} = (A_0 + \varepsilon A_1)u + F\big(u(t-T)\big),$$

where $u = (u_1, u_2) \in R^2$, A_0 and A_1 are 2×2 matrices, $T > 0$ is delay time.

The main assumptions are that the parameter ε is positive and sufficiently small: $0 < \varepsilon \ll 1$ and that the real parts of the eigenvalues of the matrix A_0 have zero real parts.

As the phase space of system we fix $C_{[-T,0]}(R^2)$.

The nonlinear function F has a compact support. We consider two particularly interesting classes of functions.

Firstly, we consider the function F depending on the scalar product v of the some fixed vector h such that $||h|| = 1$ by the vector u (i.e. $v = (h, u)$). In this case function F has compact support since F is equal to zero, if v is outside a segment $[-p, p]$:

$$F(u) = \begin{cases} f(v) & \text{if } |v| \leqslant p, \\ 0, & \text{if } |v| > p, \end{cases}$$

where $p > 0$ is some fixed constant. We assume that the function $f(v)$ is piecewise smooth.

Secondly, we consider another function having compact support:

$$F(u) = \begin{cases} f(u_1)\Phi(u_2) & \text{if } |u_1| \leqslant p, \\ 0 & \text{if } |u_1| > p. \end{cases}$$

We study the questions of existence and stability of periodic solutions in considered model for sufficiently small values of ε.

We construct nonlocal cycles with an amplitude of order unity or with an asymptotically large amplitude.

It is shown that the dynamical properties of the considered model sufficiently depend on the definitions of compactly supported nonlinear functions.

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DYNAMICS OF THE EQUATION WITH IMPULSE DELAYED FEEDBACK

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This talk discusses the dynamics of a second-order differential equation with impulsive delayed feedback:

$$\ddot{x} + \sigma\dot{x} + x = f(x(t-h)),$$

where $\sigma > 0$ and $h > 0$. Regarding the nonlinear function $f(x)$, we assume that it is of impulse type, i.e., $f(x) = 0$, for $x \neq 0$ and $\int\limits_{-a}^{b} f(x)dx = f_0$ for any $a, b > 0$.

This problem can be considered as a limiting case of the problem

$$\ddot{x} + \sigma\dot{x} + x = \lambda F(x(t-h)),$$

where $\lambda \gg 1$, and the function F takes nonzero values only on the interval $[-1, 1]$.

First, we define a class of initial conditions $S(A)$ that depends on a real parameter A and consists of functions $\varphi(t) \in \tilde{C}^1$ defined on the interval $[-h, 0]$, for which $\varphi(t) \neq 0$ for $t < 0$, $\varphi(0) = 0$, and $\dot{\varphi}(0) = A$. For each initial function from $S(A)$, we construct a solution $x_A(t)$ using the stepwise method and find its first positive root $t_* = t_*(A) :$ $x_A(t_*(A)) = 0$. If the condition $t_*(A) > h$ is satisfied, we can assert that $x_A(t_*(A) + t) \in S(\overline{A})$, where $\overline{A} = p(A) = \dot{x}_A(t_*(A))$. This defines the mapping $A_{n+1} = p(A_n)$, the dynamics of which describes the behavior of solutions of the original differential equation with delay. The specific form of the mapping and its properties depend on the parameters σ, f_0, h.

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IDEMPOTENT COMPATIBLE MAPS AND DISCRETE INTEGRABLE SYSTEMS ON THE TRIANGULAR LATTICE

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In this talk we will present three equivalence classes of rational non-invertible multidimensional compatible maps. These maps turns out to be idempotent and by construction they admit birational partial inverses (companion maps) which are Yang-Baxter maps. The maps in question can be reinterpreted as systems of difference equations defined on the edges of the $\mathbb{Z}^2$ graph. Finally, we associate these compatible systems of difference equations with integrable difference equations defined on the triangular lattice $Q(A2)$.

CASCADES OF LORENZ ATTRACTORS NEAR THE FIRST FOLIATION TANGENCY CURVE

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The Lorenz attractor is the first example of a robustly chaotic non-hyperbolic attractor. Each orbit of such an attractor has a positive top Lyapunov exponent, and this property persists under small perturbations despite possible bifurcations of the attractor. In this talk, we describe the boundary of the Lorenz attractor existence region in the Shimizu-Morioka model. As in the classical Lorenz system, a part of the boundary is associated with the curve $l_{A=0}$, where the first tangency between some Lyapunov subspaces occurs along orbits of the attractor. However, in the Lorenz system, the curve $l_{A=0}$ forms the exact boundary of the Lorenz attractor existence region. Beyond this curve, the attractor is not robustly chaotic, although it may be indistinguishable from the Lorenz attractor in simple numerical experiments. In the Shimizu-Morioka model, the curve $l_{A=0}$ is divided into two parts. The Lorenz attractor existence region adjoins $l_{A=0}$ along the first part of this curve, as in the Lorenz system. Near the second part, as we show, the region of the existence of the Lorenz attractor is fractal. We describe two infinite cascades of disjoint subregions with the Lorenz attractor. One cascade occurs along the curve $l_{A=0}$, another – in the transversal direction. Based on the analysis of the 1D Lorenz maps, we show that along the cascades, the Lorenz attractor undergoes "doubling bifurcations", leading to a complication of its topological structure.

This is a joint work with V. Koryakin, K. Safonov, and A. Shilnikov. The work is supported by the RSF grant No. 25-11-20069.

ON THE RESONANT PERIODIC MOTIONS OF A NEARLY AUTONOMOUS HAMILTONIAN SYSTEM IN ZERO FREQUENCY CASES OF THE LIMITING AUTONOMOUS PROBLEM

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The report addresses the motions of a nearly autonomous, 2π-periodic in time Hamiltonian system with two degrees of freedom in the neighborhood of a trivial equilibrium. The values of the parameters near the boundary of the stability region of this equilibrium that corresponds to the zero frequency case in the limiting autonomous problem are considered. The cases are distinguished when the other frequency is non-resonant (not equal to an integer or half-integer number) and resonant, i.e. a multiple parametric resonance is realized in the system. The stability and instability regions (parametric resonance regions) of the trivial equilibrium of the system are obtained. The question of the existence in its neighborhood of analytic (in integer or fractional powers of a small parameter) resonant periodic motions, their number and stability in the linear approximation is resolved. Earlier [1], similar results for the studied cases of multiple parametric resonances have been obtained in the sections of the parameter space corresponding to a fixed (resonant) value of one of the parameters. In this report, complete neighborhoods of resonance points in the parameter space are considered. As expected, this leads to complication of both the stability diagram of the trivial equilibrium and the distribution and nature of the stability of the periodic solutions under study. For each resonance case, an algorithm for analyzing these motions is developed. Differences and similarities in the properties of the system for different resonant cases are established, parallels are found with multiple resonance cases of another type, arising in the systems with another structure of the perturbing part of the Hamiltonian function.

As an application to the obtained general theoretical results, resonant periodic motions of a dynamically symmetric satellite (rigid body) in a central Newtonian gravitational field on a weakly elliptical orbit in the vicinity of its stationary rotation (cylindrical precession) are constructed.

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ON SOLUTIONS OF THE EULER EQUATION FOR INCOHERENT FLUID IN CURVED SPACES

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Hodograph equations for the Euler equation for compressible fluid with constant pressure in curved spaces are discussed. It is shown that the use of known results concerning geodesics and associated integrals allow us to construct several types of hodograph equations. These hodograph equations provide us with various classes of solutions of the Euler equation parametrized by arbitrary functions including stationary solutions. Particular cases of cone and sphere in the 3-dimensional Euclidean space are analysed in detail. The Euler equation on the sphere in the 4-dimensional Euclidean space is considered too.

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PIECEWISE SMOOTH SOLUTIONS OF A SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS IN A SIMPLE CRITICAL CASE

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Consider the system of integro-differential equations

$$\frac{\partial u}{\partial t} = (A_0+\varepsilon A_1)u+F_2(u,u)+F_3(u,u,u)+(D_0+\varepsilon D_1)\frac{1}{2\pi}\int_0^{2\pi} u(t,x+s)ds$$

with periodic boundary condition

$$u(t,x+2\pi) \equiv u(t,x),$$

where $u = u(t,x) \in \mathbb{R}^n$, A_0, A_1, D_0, D_1 are $n \times n$ matrices, $F_2(\cdot,\cdot)$, $F_3(\cdot,\cdot,\cdot)$ are linear functions by each argument in $\mathbb{R}^n$.

In this work we consider the case when matrix A_0 has a simple pair of conjugate purely imaginary eigenvalues $\lambda = \pm i\omega$. In this case, the solution of the boundary value problem is represented as an asymptotic series

$$\begin{aligned} u(t,x) = &\varepsilon^{1/2}(\xi(\tau,x)a\exp(i\omega t) + \bar{\xi}(\tau,x)\exp(-i\omega t))+ \\ &+ \varepsilon u_2(t,\tau,x) + \varepsilon^{3/2}u_3(t,\tau,x) + \dots, \end{aligned}$$

where $\tau = \varepsilon t$, $\xi(\tau,x)$ is scalar 2π-periodic over x function, $u_2(t,\tau,x)$, $u_3(t,\tau,x)$ are 2π-periodic over x vector functions.

Let us substitute this series into the original system and equate the terms with the same degrees of ε. We obtain with $\varepsilon^{3/2}$ the boundary value problem with unknown $\xi(\tau,x)$

$$\frac{\partial \xi}{\partial \tau} = \lambda\xi + \sigma(\xi|\xi|^2 - M(\xi|\xi|^2)) + \beta\xi M(|\xi|^2) + \gamma\bar{\xi}M(\xi^2),$$

$$\xi(\tau,x+2\pi) \equiv \xi(\tau,x), \quad M(\xi) = 0,$$

where $M(\xi) = \frac{1}{2\pi}\int_0^{2\pi} \xi(\tau,x)dx$.

The study of this boundary value problem is based on the methods presented in the [1], [2]. This boundary value problem has stepwise solutions with respect to x periodic with respect to τ.

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CONTRAST STRUCTURES IN A REACTION-DIFFUSION SYSTEM WITH MULTISCALE DIFFUSION COEFFICIENTS AND DISCONTINUOUS REACTION FUNCTIONS

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The paper studies a 1D reaction-diffusion system with multiscale diffusion coefficients, modeling interactions of substances with distinct kinetics (e.g., chemical reactions in porous media or CO_2 storage). A key feature is discontinuities (1st kind) in reaction terms, leading to sharp concentration gradients. Singular perturbation for the fast component creates a transition layer with rapid concentration changes. Analytical results validate usage of numerical methods for such systems (e.g., [1,2]). The mathematical formulation of the problem is defined as follows:

$$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \begin{cases} f^{(-)}(u, v, x, \varepsilon), \\ f^{(+)}(u, v, x, \varepsilon), \end{cases}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} + \begin{cases} g^{(-)}(u, v, x, \varepsilon), x \in (-1, 0), \\ g^{(+)}(u, v, x, \varepsilon), x \in (0, 1), \end{cases} \qquad t > 0,$$

$$\frac{\partial u}{\partial x}(-1, t) = \frac{\partial u}{\partial x}(1, t) = 0, \quad \frac{\partial v}{\partial x}(-1, t) = \frac{\partial v}{\partial x}(1, t) = 0, \quad t > 0,$$

$\varepsilon > 0$ — small parameter, f and g suffers a gap of the first kind at the point $x = 0$.

Based on the asymptotic approximation of the stationary solution of the problem and the upper and lower solutions constructed in the work and [3], it is proved that under conditions (A1)-(A3), stated below, and a sufficiently small $\varepsilon > 0$, a stationary solution to the problem exists, is locally unique and asymptotically Lyapunov stable in the constructed domain.

(A1) There is a solution $\overline{v}^{(\mp)}(x)$, $\overline{u}^{(\mp)}(x) = \varphi^{(\mp)}(\overline{v}^{(\mp)}(x), x)$ of the problem:

$$0 = f^{(\mp)}(u^{(\mp)}, v^{(\mp)}, x, 0),$$
$$\frac{d^2 v^{(\mp)}}{dx^2} = g^{(\mp)}(u^{(\mp)}, v^{(\mp)}, x, 0),\ \frac{dv^{(\mp)}}{dx}(\mp 1) = 0,\ v^{(\mp)}(0) = \psi_v.$$

(A2) There are such ψ_{0v} and ψ_{0u} that for the functions:

$$J(\psi_v) = \frac{d\overline{v}^{(-)}}{dx}(0, \psi_v) - \frac{d\overline{v}^{(+)}}{dx}(0, \psi_v),$$
$$H(\psi_u, \psi_v) = \sqrt{2 \int\limits_{\varphi^{(-)}(\psi_v, 0)}^{\psi_u} f^{(-)}(u, \psi_v, 0, 0)du -}$$
$$- \sqrt{2 \int\limits_{\varphi^{(+)}(\psi_v, 0)}^{\psi_u} f^{(+)}(u, \psi_v, 0, 0)du},$$

conditions are met: $J(\psi_{0v}) = H(\psi_{0u}, \psi_{0v}) = 0$, $\dfrac{dJ}{d\psi_v}(\psi_{0v}) > 0$, $\dfrac{\partial H}{\partial \psi_u}(\psi_{0u}, \psi_{0v}) > 0$.

(A3) Let the inequality be fulfilled:

$$g_v^{(\mp)}(\overline{u}^{(\mp)}(x), \overline{v}^{(\mp)}(x), x, 0) -$$
$$- |\varphi_v^{(\mp)}(\overline{v}^{(\mp)}(x), x) g_u^{(\mp)}(\overline{u}^{(\mp)}(x), \overline{v}^{(\mp)}(x), x, 0)| > -\lambda_0^{(\mp)},$$

$\lambda_0^{(\mp)}$ — the main eigenvalues of: $\dfrac{d^2\Psi^{(\mp)}}{dx^2} + \lambda^{(\mp)}\Psi^{(\mp)} = 0, \Psi^{(\mp)}(0) = \dfrac{d\Psi^{(\mp)}}{dx}(\mp 1) = 0.$

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CONVECTIVE CAHN-HILLIARD EQUATION IN THE CASE OF TWO SPATIAL INDEPENDENT VARIABLES. PERIODIC TRAVELING WAVES

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Let us consider the Cahn-Hilliard equation in its natural form, when the unknown function depends on two spatial variables x_1, x_2

$$u_t + \Delta^2 u + \Delta(a_1 u + a_2 u^2 + a_3 u^3) + c_1(u^2)_{x_1} + c_2(u^2)_{x_2} = 0, \qquad (1)$$

where Δ is the Laplace operator, $a_1, a_2, c_2, c_2 \in \mathbb{R}$. If $c_1 = c_2 = 0$, then we obtain the original version of the Cahn-Hilliard equation [1]. If $c_1^2 + c_2^2 \neq 0$, then equation (1) is usually called the convective Cahn-Hilliard equation [2,3].

Let equation (1) be supplemented with periodic boundary conditions

$$u(t, x_1 + 2l_1, x_2) = u(t, x_1, x_2 + 2l_2) = u(t, x_1, x_2), \qquad (2)$$

where $l_1, l_2 \in \mathbb{R}_+$. In the report, we will limit ourselves to the analysis of the boundary conditions (2) in the case when $l_1 = l_2 = \pi$.

Sufficient conditions for the existence of wave solutions (traveling periodic waves) for the boundary value problem (BVP) (1),(2) are obtained in the case when $c_1^2 + c_2^2 \neq 0$, i.e. solutions that have the structure of traveling wave

$$u = F(n_1 x_1 + n_2 x_2 + \omega(n_1, n_2)t), \qquad (3)$$

where $F(y)$ is a sufficiently smooth 2π – periodic function of the variable y. Here $n_1, n_2 \in \mathbb{Z}$ the set of integers with $n_1^2 + n_2^2 \neq 0$.

In the case $c_1 = c_2 = 0$, instead of traveling waves, we obtain the equilibrium states of the studied BVP, i.e. solutions of the following form

$$u = F(y),\ y = n_1x_1 + n_2x_2. \tag{4}$$

It is shown that in a general situation, solutions (3) ((4)) form two-dimensional invariant manifolds $V_2\,(dimV_2 = 2)$. The question of the stability of these two-dimensional manifolds, as well as the question of the stability of the solutions forming them, are studied. Asymptotic formulas for the solutions forming two-dimensional invariant manifolds are obtained.

In the analysis of the BVP under study, the results and methodology of the work [3,4] were used.

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CONFORMAL INVERSION IN A MINKOWSKI PHASE SPACE: RECURSION OPERATOR AND QUADRATIC ALGEBRA

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Using the conformal inversion $\mathcal{R}$, we study the Hamiltonian dynamics for a free particle in a Minkowski phase space. We define the corresponding Hamiltonian function and Poisson bracket, and derive the Hamiltonian vector fields describing the system dynamics. Furthermore, using the Hamilton-Jacobi separability, we construct associated recursion operators for Hamiltonian vector fields, and deduce related integrals of motion. We infer the existence of a quadratic algebra generated by integrals of motion.

STATIONARITY OF THE FRONT IN THE REACTION-DIFFUSION PROBLEM IN THE CASE OF A PIECEWISE-LINEAR DISCONTINUOUS SOURCE OF MODULAR TYPE

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We consider the boundary value problem for the reaction-diffusion equation with a piecewise-linear discontinuous source of modular type:

$$\begin{aligned}
&\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\varepsilon\frac{\partial u}{\partial t}=f(u,x,\varepsilon),\ x\in(-1,1),\ t\in(0,T],\\
&\frac{\partial u}{\partial x}(-1,t,\varepsilon)=0,\ \frac{\partial u}{\partial x}(1,t,\varepsilon)=0,\ t\in(0,T],\\
&u(x,0,\varepsilon)=u_{init}(x,\varepsilon),\ x\in[-1,1],
\end{aligned}\qquad(1)$$

where

$$f(u,x,\varepsilon)=\begin{cases}u-\varphi^{(-)}(x), & u<0,\\ u-\varphi^{(+)}(x), & u\ge 0.\end{cases}$$

Here $\varepsilon\in(0;\varepsilon_0]$ is a small parameter and $u_{init}(x,\varepsilon)$ is a continuous function of a front form.

The front is formed in the vicinity of a point, the law of motion of which we denote as $\hat{x}(t)$. This point is determined by the condition $u(\hat{x}(t),t)=0$.

Definition. The function $u(x,t)$ from the class

$$C^{1,0}([-1,1]\times[0,T])\bigcap C^{1,1}([-1,1]\times(0,T])\bigcap C^{2,1}((-1,\hat{x})\times(0,T]\bigcup \bigcup(\hat{x},1)\times(0,T]).$$

we will call the solution to problem (1) if it satisfies the equation (1) in each domain $(-1,\hat{x})\times(0,T]$ and $(\hat{x},1)\times(0,T]$ and the initial and boundary conditions.

If there is a stationary solution to problem (1), then it is a solution to the following problem:

$$\varepsilon^2 \frac{\partial^2 u_s}{\partial x^2} = f(u_s(x), \hat{x}_s, \varepsilon), \ x \in (-1, 1),$$
$$\frac{\partial u_s}{\partial x}(-1, \varepsilon) = 0, \ \frac{\partial u_s}{\partial x}(1, \varepsilon) = 0. \qquad (2)$$

The paper obtains sufficient conditions for the existence, uniqueness and asymptotic stability according to Lyapunov of a stationary solution of the initial-boundary value problem (1) sa a solution to problem (2). The corresponding theorem is proved, as well as a theorem on stabilization of the solution of a parabolic problem to a unique stationary solution. To illustrate the stabilization process, the results of numerical implementation of the solution for given parameters are given. The validity of the numerical calculations is confirmed by the proven theorem.

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OPTIMAL CONTROL OF THE MOVEMENT OF AN UNMANNED AERIAL VEHICLE IN A VARIABLE EXTERNAL ENVIRONMENT

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This work investigates optimal control problems for unmanned aerial vehicles (UAVs) operating in three-dimensional coordinate systems. The mathematical model considers a UAV moving along a trajectory $q(t) = (x(t), y(t), z(t))$ in the coordinate system $OXYZ$. It is assumed that the UAV, starting at time t_0 from initial position q_0 with initial velocity $\dot{q}_0$, must reach a target position q_T with specified final velocity $\dot{q}_T$ within time $T > 0$. The motion equations for the UAV can be written as:

$$m\ddot{q} + f(q, \dot{q}, t) = U(t), \quad t_0 < t < t_1 = t_0 + T, \tag{1}$$

with boundary conditions:

$$q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0, \quad q(t_1) = q_1, \quad \dot{q}(t_1) = \dot{q}_1, \tag{2}$$

where m is the UAV mass, $f(q, \dot{q}, t)$ represents nonlinear forces acting on the UAV (including aerodynamic forces, gravitational forces, and other external disturbances), $U(t) = (U_1(t), U_2(t), U_3(t))$ is the control vector, and $|q| = (q, q)^{1/2} = (x^2 + y^2 + z^2)^{1/2}$ is the Euclidean norm of the position vector, where $(q_1, q_2) = x_1x_2 + y_1y_2 + z_1z_2$ is the scalar product in $\mathbb{R}^3$. The specific form of the force function $f(q, \dot{q}, t)$ depends on the particular flight conditions. It is assumed that the function $f(q, \dot{q}, t)$ is continuously differentiable and satisfies the condition $f_j(0, 0, t) \equiv 0$ for all control components $U_j(t)$.

The objective functional for the optimization problem is defined as:

$$J(U(t)) = \int_0^T (U(t), U(t))dt = \int_0^T (U_1^2(t) + U_2^2(t) + U_3^2(t))dt. \tag{3}$$

Two optimal control problems are formulated:

Problem 1. Find the control $U(t)$ that satisfies the differential equation with boundary conditions and minimizes the functional $J(U)$.

Problem 2. Find the control $U(t)$ such that $J(U) \leq L$ for given $L > 0$, satisfying the boundary conditions in minimum time $T > 0$.

For solving these problems, variational methods of optimal control theory are employed, as developed in the referenced work [1].

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ON ONE SOLUTION OF THE PROBLEM OF OSCILLATIONS OF MECHANICAL SYSTEMS WITH MOVING BOUNDARIES

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The problem of oscillations of bodies with moving boundaries, formulated as a differential equation with boundary and initial conditions, is a non-classical generalization of the hyperbolic problem. To facilitate the construction of a solution to this problem and to justify the choice of the solution type, equivalent integro-differential equations with symmetric and non-stationary kernels and non-stationary integration limits are constructed. The advantages of the integro-differential equation method are revealed when moving to more complex dynamic systems carrying concentrated masses oscillating under the action of moving loads. The method is extended to a wider class of model boundary value problems that take into account bending rigidity, resistance of the external environment, and the rigidity of the base of the oscillating object. The solution is given in dimensionless variables accurate to the values of the second order of smallness of relatively small parameters characterizing the velocity of the boundary. An approximate solution is found to the problem of transverse oscillations of a viscoelastic beam with bending rigidity, taking into account the action of damping forces.

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CLASSIFICATION OF 4D-ORBIT LIE ALGEBRAS: SEMIDIRECT SUMS OF SEMISIMPLE AND SOLVABLE

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An important example of a Poisson manifold is an affine space Π with a tensor field π^{ij} defined on it satisfying the properties:

$$\pi^{jk}(x) = -\pi^{kj}(x),$$

$$\pi^{jk}\partial_k\pi^{lm} + \pi^{lk}\partial_k\pi^{mj} + \pi^{mk}\partial_k\pi^{jl} = 0.$$

The simplest example of such a tensor is a tensor, with constant components. The study of the properties of such a Poisson manifold is reduced to simple linear algebra problems. The next important example of such a tensor is a tensor that linearly depends on coordinates x^k on Π:

$$\pi^{ij} = c_k^{ij} x^k. \tag{1}$$

The conditions for the tensor c_k^{ij} are then rewritten as follows:

$$c_k^{ij} = -c_k^{ji},$$

$$c_e^{ad}c_d^{bc} + c_e^{bd}c_d^{ca} + c_e^{cd}c_d^{ab} = 0.$$

This shows that any linear Poisson structure can be interpreted as a tensor field on the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with structure constants as given in formula (1).

This approach is considered, for example, in the book by V. I. Arnold [1]. The symplectic fibers of a Poisson manifold like this coincide with the coadjoint orbits. In other words, a mechanical system with n degrees of freedom can be described by a Lie algebra whose general-position coadjoint orbits have dimension $2n$. This fact generates interest in constructing Lie algebras with specific orbit dimensions, classifying them, and studying the properties of the resulting integrable systems.

The simplest case among integrable systems is systems with one degree of freedom. They are modeled by Lie algebras, in which the coadjoint generic orbits are two-dimensional. A complete classification of such Lie algebras is carried out in [2], which also provides references to reports on specific integrable systems corresponding to these algebras. In the report [3] the topology of generic coadjoint orbits is studied in algebras from the list of the main theorem from [2].

This report is devoted to the next important case of integrable systems with two degrees of freedom. Such systems are modeled by Lie algebras with four-dimensional coadjoint generic orbits. A complete classification of such Lie algebras can be useful for solving various problems in the theory of integrable systems, for example, for verifying the generalized Mishchenko–Fomenko conjecture formulated in [4], or for constructing geodesic flows in coadjoint orbits using the construction proposed in [5]. In particular, [6] considers examples of Lie algebras with two-dimensional and four-dimensional coadjoint orbits with rational invariants in the context of a construction from [5].

The main result presented in the report:

Theorem.[7] Let $\mathfrak{g}$ be a real algebra represented as a semidirect sum $\mathfrak{s} +_\varphi \mathfrak{h}$, where $\mathfrak{s}$ is a nontrivial semisimple Lie algebra and $\mathfrak{h}$ is a solvable ideal. And let The generic coadjoint orbits have dimension 4. Then algebra $\mathfrak{g}$ is isomorphic to one of the algebras:

- a direct sum of a semisimple and solvable algebra with two-dimensional coadjoint generic orbits;
- semidirect sum of $sl(2, \mathbb{R}) +_\varphi \mathbb{R}^2$ according to the standard representation;

- semidirect sum of $sl(2,\mathbb{R}) +_\varphi \mathbb{R}^3$ according to the adjoint representation;
- semidirect sum of $so(3,\mathbb{R}) +_\varphi \mathbb{R}^3$ according to the standard three-dimensional representation.

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FRONT FORMATION IN THE REACTION-DIFFUSION PROBLEM WITH NONLINEAR DIFFUSION

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In [1], the existence of a moving front solution was proved for the following reaction-diffusion problem with nonlinear diffusion:

$$\varepsilon^2 \frac{\partial}{\partial x}\left(D(u,x,\varepsilon)\frac{\partial u}{\partial x}\right) - \varepsilon\frac{\partial u}{\partial t} - f(u,x,\varepsilon) = 0, \quad x \in (-1,1), \quad t \in (0,T],$$
$$\frac{\partial u}{\partial x}(-1,t,\varepsilon) = 0, \quad \frac{\partial u}{\partial x}(1,t,\varepsilon) = 0, \qquad t \in [0,T],$$
$$u(x,0,\varepsilon) = u_{init}(x,\varepsilon), \qquad x \in [-1,1].$$

It was assumed that the front had already formed at the initial moment of time (i.e., the u_{init} function has the form of a sharp inner transition layer) and the process of its evolution over time was considered.

Here we are discussing the very problem of the front formation. Earlier in [2, 3] this problem has been investigated in some other cases, and in this paper we intended to solve it for the reaction-diffusion problem with nonlinear diffusion.

Sufficient conditions have been formulated to ensure the formation of a sharp inner transition layer from the initial conditions of quiet general form, and an estimate of the duration of the transition process has been obtained.

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SMOOTHNESS OF ONE-DIMENSIONAL FACTOR MAP IN SYSTEMS WITH THE LORENZ ATTRACTOR

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Since the first theoretical studies of the Lorenz attractor, one-dimensional maps have been used to describe the topological structure and bifurcations of the attractor. The reduction to the one-dimensional maps is possible due to the existence of a strongly stable invariant foliation in a neighborhood of the attractor. However, in the general case, the strongly stable invariant foliation has very low smoothness, since the attractor contains a saddle singularity. This fact causes various technical difficulties in studying the Lorenz attractor and does not to apply many results from one-dimensional dynamics.

In this talk, we give verifiable sufficient conditions that provide the existence of smooth invariant foliation for systems satisfying the Shilnikov criteria [1]. Namely, we consider a system undergoing one of the codimension-2 bifurcations of a homoclinic butterfly: inclination flip, orbit flip, or bifurcation with a neutral saddle. It is known that these bifurcations lie on the boundary of existence of the Lorenz attractor. We show that for systems close to these homoclinic bifurcation, the smoothness of the strongly stable invariant foliation is defined by the gap between the leading stable eigenvalue of the saddle and the other stable eigenvalues. Our sufficient conditions imply that the foliation can have arbitrarily finite smoothness, if the gap is large enough. We also obtain improved estimates on the equation for the one-dimensional factor map and discuss its renormalization properties.

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DEGENERATE HAMILTONIAN OPERATORS, FLAT PENCILS AND SUPERINTEGRABLE SYSTEMS

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In [1], B. A. Dubrovin and S. P. Novikov studied for the first time Hamiltonian structures for PDEs whose operator is homogeneous in the degree of derivation. In particular they showed that in the case of non-degenerate leading term, a given first-order homogeneous operator satisfies the Hamiltonian property if and only if its coefficients define (a) a flat metric and (b) the Christoffel symbols of the corresponding Levi-Civita connection.

In this talk, we present a partial geometric interpretation of these operators in the general case (without the non-singularity assumption) in terms of a symmetric $(2,0)$ tensor and a compatible contravariant normal connection. This interpretation leads to a deeper understanding of pair of Hamiltonian operators which are degenerate and the related pencil of flat (almost-) metrics. We finally show some applications to the classification of superintegrable systems under the abundancy property.

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ANALYSIS OF OSCILLATIONS NEAR TRIANGULAR LIBRATION POINTS AT THIRD-ORDER RESONANCE

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This report addresses the spatial restricted elliptic problem of three bodies (material points) gravitating toward each other under Newton's law of gravitation. The eccentricity of the orbit of the main attracting bodies is assumed to be small, and nonlinear oscillations of a passively gravitating body near a Lagrangian triangular libration point are studied. It is assumed that in the limiting case of the circular problem the ratio of the frequency of rotation of the main bodies about their common center of mass to the value of one of the frequencies of small linear oscillations of the passive body is exactly equal to three. A detailed analysis is made of two different particular cases of influence of the three-dimensionality of the problem on the characteristics of nonlinear oscillations of the passive body.

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LOCAL DYNAMICS OF A SECOND-ORDER EQUATION WITH TWO DELAYS AT THE DERIVATIVE

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Consider a second-order differential equation with two delayed feedbacks (1):

$$\varepsilon\ddot{x} + \dot{x} + \delta x = F(\dot{x}(t-T)) + G(\dot{x}(t-\tau)), \tag{1}$$

Here $\varepsilon \ll 1, \delta \ll 1$ are small and proportional parameters $\delta = k\varepsilon$, T and τ is a delay parameter, real and positive. The functions F and G are sufficiently smooth, such that $G(0) = 0$ and $F(0) = 0$.Thus, equation (1) has a zero equilibrium state. Note, that the problem under consideration is singularly perturbed.

Let's make a time substitution (1) $t \to Tt$ and $x(Tt) \to x(t)$. We arrive at an equivalent equation (2):

$$\varepsilon\ddot{x} + \dot{x} + k\varepsilon x = F(\dot{x}(t-1)) + G(\dot{x}(t-\tau)), \tag{2}$$

Note that the delay τ is small $\tau = \varepsilon h$:

$$\varepsilon\ddot{x} + \dot{x} + k\varepsilon x = F(\dot{x}(t-1)) + G(\dot{x}(t-\varepsilon h)). \tag{3}$$

Let study the local dynamics in the vicinity of the equilibrium state,in the phase space $C^1_{[1,0]}$.

The characteristic quasi-polynomial of the linearized at zero equation (3) has the form:

$$\varepsilon\lambda^2 + \lambda + k\varepsilon = b\lambda e^{-\lambda} + c\lambda e^{-\varepsilon h\lambda}. \tag{4}$$

Let's select the stability region by constructing a D-split based on the parameters b and c. To do this, let's represent $\lambda = i\omega$ and obtain the equation (5):

$$-\varepsilon\omega^2 + i\omega + k\varepsilon = bi\omega e^{-i\omega} + ci\omega e^{-\varepsilon hi\omega}. \tag{5}$$

Let's separate the real and imaginary parts from the equation (5):

$$\begin{cases} b = \dfrac{1 - c\cos(\varepsilon h\omega)}{\cos(\omega)}. \\ c = \dfrac{(-\varepsilon\omega^2 + k\varepsilon)\cos(\omega) - \omega\sin(\omega)}{\omega sin(\varepsilon h\omega)\cos(\omega) - \cos(\varepsilon h\omega)\omega\sin(\omega)} \end{cases}. \tag{6}$$

However, for sufficiently small values of ε, it is possible to construct an asymptotic approximation of the boundary of the stability region. These are the limiting boundaries of the stability region as $\varepsilon \to 0$.

$$\lambda = \frac{\delta}{\varepsilon}i + \Theta(\varepsilon)i + \Omega i + 2\pi n i + ... \tag{7}$$

Here $\frac{\delta}{\varepsilon}$ is a rapidly varying variable, $\Theta(\varepsilon)$ such that $\frac{\delta}{\varepsilon} + \Theta(\varepsilon)$ is a multiple of 2π. Substitute the replacement (7) into the equation (4) and discard the small parts:

$$-\frac{\delta^2}{\varepsilon} + \frac{\delta}{\varepsilon}i + O(\varepsilon) = b\frac{\delta}{\varepsilon}ie^{-i\Omega} + c\frac{\delta}{\varepsilon}ie^{-\delta hi}. \tag{8}$$

Simplify the equation (8):

$$e^{-i\Omega} = \frac{-\delta + i - cie^{-\delta hi}}{bi}. \tag{9}$$

$$|-\delta + i - cie^{-\delta hi}| = |bi|. \tag{10}$$

When $\delta = 0$:

$$|i - ci| = |bi|. \tag{11}$$

When $\delta \neq 0$:

$$|-\delta + i - cie^{-\delta hi}| = |bi|. \tag{12}$$

The study was carried out with the financial from the Government of the Yaroslavl Region as part of a scientific project No. 15NP/2024.

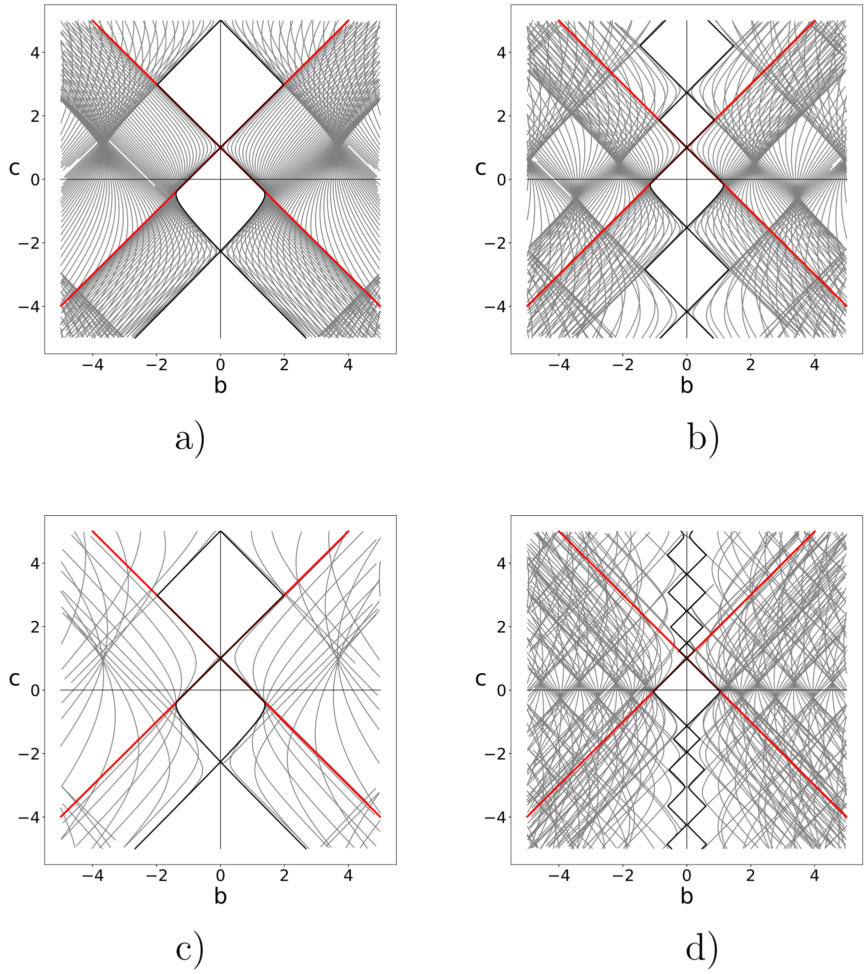

Fig. 1: Figure of D-partitions red lines are straight lines from equation (11), gray lines are the solution of the system of equations (6), and black lines are the limiting curves from equation (12). Parameter values: a) $h = 1$, $\varepsilon = 0.01$, $k = 2$; b) $h = 2$, $\varepsilon = 0.01$, $k = 2$; c) $h = 1$, $\varepsilon = 0.05$, $k = 2$; d) $h = 5$, $\varepsilon = 0.01$, $k = 2$.

CHAOTIC DYNAMICS OF A DELAYED PULSE FEEDBACK DIFFERENTIAL EQUATION

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Consider a second-order nonlinear differential equation with a delay

$$\ddot{x} + \sigma\dot{x} + x = f(x(t-h)), \tag{1}$$

where $\sigma > 0$ and $h > 0$. With respect to the nonlinear function $f(x)$, we assume that it has an impulse type: $f(x) = 0$, for $x \neq 0$ and $\int\limits_{-a}^{b} f(x)dx = f_0$ for any $a, b > 0$. Let study the behavior of solutions for fixed values of the parameters σ, f_0, h.

The study of the original problem (1) can be reduced to the study of the mapping $p : \Omega \to \Omega$, where Ω is the set of all sets $\Gamma = \{(t_0, A_0), ..., (t_n, A_n) | t_0 = 0, t_i > t_{i+1}\}$. For some cases where explicit mapping formulas exist, $p(\Gamma)$. For example, for $\Gamma = \{(0, A)\}$ and $\sigma > 2$, the mapping is $p(\Gamma) = p(A)$

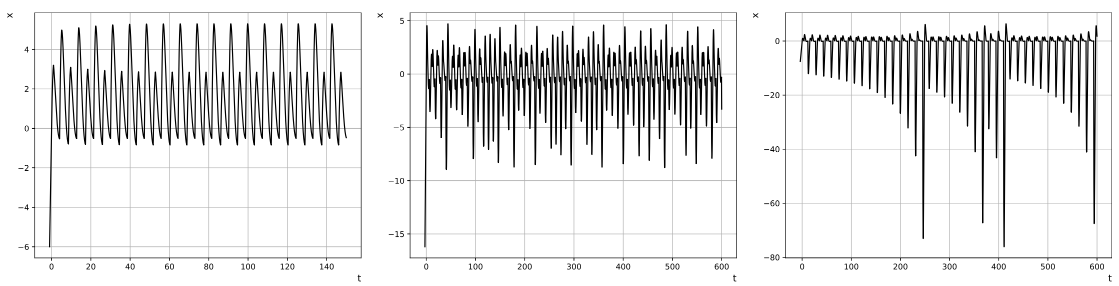
$$p(A) = \lambda_1 \left(\frac{A^2 + f_0 e^{-\lambda_1 h}}{A(\lambda_1 - \lambda_2)} \right) \left(\frac{A^2 + f_0 e^{-\lambda_2 h}}{A^2 + f_0 e^{-\lambda_1 h}} \right)^{\frac{\lambda_1}{\lambda_1 - \lambda_2}} - \\ - \lambda_2 \left(\frac{A^2 + f_0 e^{-\lambda_2 h}}{A(\lambda_1 - \lambda_2)} \right) \left(\frac{A^2 + f_0 e^{-\lambda_2 h}}{A^2 + f_0 e^{-\lambda_1 h}} \right)^{\frac{\lambda_2}{\lambda_1 - \lambda_2}}.$$

Theorem

If $\sigma > 2$, then there are periodic repulsive solutions or solutions at $t \geq T$ tending to zero.

If $0 < \sigma < 2$, then the solutions tend to have a stable periodic solution or have chaotic fluctuations.

The simple behavior of the functions is shown in Fig. 2 (a), where the solution tends to zero or a stable periodic regime. The complex behavior of the functions is shown in Fig. 2 (b,c), where the solution performs chaotic fluctuations.



(a) (b) (c)

Fig. 2: Graphs of the solution of the equation (1) for:
a) $h = 1, \sigma = 1, f_0 = -10, A = 6$,
b) $h = 2.7, \sigma = 0.4, f_0 = 10, A = 6$,
c) $h = 3.8, \sigma = 1.24, f_0 = 10, A = 2$

This work was carried out within the framework of a development programme for the Regional Scientific and Educational Mathematical Center of the Yaroslavl State University with financial support from the Ministry of Science and Higher Education of the Russian Federation (Agreement on provision of subsidy from the federal budget No. 075-02-2025-1636).

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R-MATRIX VALUED LAX PAIR FOR ELLIPTIC CALOGERO-INOZEMTSEV SYSTEM AND ASSOCIATIVE YANG-BAXTER EQUATIONS OF BC_n TYPE

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We consider the elliptic Calogero-Inozemtsev system of BC_n type with five arbitrary constants and propose R-matrix valued generalization for $2n \times 2n$ Takasaki's Lax pair. For this purpose we extend the Kirillov's B-type associative Yang-Baxter equations to the similar relations depending on the spectral parameters and the Planck constants. General construction uses the elliptic Shibukawa-Ueno R-operator and the Komori-Hikami K-operators satisfying reflection equation. Then, using the Felder-Pasquier construction the answer for the Lax pair is also written in terms of the Baxter's 8-vertex R-matrix. As a by-product of the constructed Lax pair we also propose BC_n type generalization for the elliptic XYZ long-range spin chain, and we present arguments pointing to its integrability.

CLASSIFICATION OF SOLITON SOLUTIONS IN GENERALIZED KORTEWEG-DE VRIES AND SCHRÖDINGER EQUATIONS

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The classical Korteweg–de Vries equation is typically used to describe long waves in shallow water. However, when the nonlinear term is generalized, this model finds applications in a vast number of physical problems [1, 2]. An interesting question is to qualitatively describe the behavior of soliton solutions, which often turn out to be stable [3]. In this work, we provide a complete classification of soliton solutions in the generalized KdV equation

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial^3 U}{\partial x^3} = 0 \tag{1}$$

and generalized Schrödinger equations

$$i\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} + F(|U|)U = 0 \tag{2}$$

Here, the function U is a scalar function of the arguments x (coordinate) and t (time), and the function F is a smooth function generalizing the nonlinearity. For KdV equation is proved that a solitonic solution $U = u(x-ct) = u(\xi)$ (u vanishes at plus or minus infinity) can be one of four possible types: classical soliton (that is, sign-constant, with one extremum and two inflection points), pyramidal soliton (with a large number of inflection points), classical kink (a monotonic solution that goes to a constant other than zero at plus (minus) infinity with one inflection point) and pyramidal kink (like the classic one, but with a lot of inflection points). Similarly, the theorem on soliton solutions in the Schrödinger equation is proved. The conditions for the various types solitons appearance for certain equations classes with nonlinearity such as a generalized polynomial or the Tchebycheff system of smooth functions are clarified. Examples of equations with a nontrivial soliton form are given.

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RICCI NILSOLITONS, EINSTEIN SOLVMANIFOLDS AND LIE ALGEBRA GRADINGS

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A Ricci soliton is a special solution of the (normalized) Ricci flow moving along the equation by diffeomorphisms, that is, where the metric $g(t)$ on a manifold M is the pull-back of the initial metric $g(0)$ by a one-parameter group of diffeomorphisms.

Consider a nilpotent Lie group $M = G$ with a left invariant Riemannian metric $g(0)$ and left-invariant RIcci solitons (nilsolitons) $g(t)$. The left-invariant metric g defines the Euclidean inner product on the Lie algebra $\mathfrak{g}$ of G. The left-invariant Ricci tensor Ric of the metric g defines a self-adjoint Ricci operator $R : \mathfrak{g} \to \mathfrak{g}$.

The problem of finding left-invariant nilsoliton metrics is equivalent to finding corresponding Einstein solvmanifolds and it can be reduced to some purely algebraic problem. More precisely, the operator R, like the identity operator Id, is not a derivation of the Lie algebra $\mathfrak{g}$, but it might happen that for some constant $c \in \mathbb{R}$ the operator $D = R - cId$ will be a derivation of the nilpotent Lie algebra $\mathfrak{g}$ and in this case the derivation D is called the *Einstein derivation* of $\mathfrak{g}$ and in this case D defines a Ricci nilsoliton or some Einstein solvmanifold.

We will discuss Einstein differentiations and their relations with positive gradings of nilpotent Lie algebras.

ASYMPTOTIC INTEGRATION OF SPATIALLY DISTRIBUTED DELAY LOGISTIC EQUATION WITH VARYING COEFFICIENT

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In this talk we study the dynamics of solutions to the linearized spatially distributed delay logistic equation

$$\frac{\partial v}{\partial t} = d\frac{\partial^2 v}{\partial x^2} + r\big[1 - v(t-1,x)\big]v, \quad x \in [0,1] \tag{1}$$

in the neighbourhood of the equilibrium $v_0 \equiv 1$. Here $v(t,x) \geq 0$ is the population size, $d > 0$ is a mobility coefficient (diffusion coefficient) and $r > 0$ is the intrinsic rate of natural increase (Malthusian coefficient), which we will assume to depend both on t and x. Finally, the presence of a time delay in the model is due to the attainment of reproductive maturity by individuals. We consider Eq. (1) together with Neumann boundary conditions

$$\left.\frac{\partial v}{\partial x}\right|_{x=0} = \left.\frac{\partial v}{\partial x}\right|_{x=1} = 0. \tag{2}$$

The spatially distributed logistic equation without delay has been studied by many authors and has quite a long history of research starting with the works of Fisher [1] and Kolmogorov et al. [2]. The delay logistic equation without spatial distribution was considered, for example, by Hutchinson in [3]. Taking in account both factors, delay and spatial distribution, results in equation with many technical difficulties. A more detailed historical overview of this issue can be found in the books [4, 5] (see also paper [6]). Equations of the form (1) with various conditions on coefficients and different types of boundary conditions are thoroughly studied in the recent monograph [7]. It also contains a list of works where the dynamics of solutions of Eq. (1) is studied.

By substituting $u = v - 1$ in (1), (2) and discarding nonlinear terms, we obtain the following linearized problem:

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} - ru(t-1,x), \quad x \in [0,1], \tag{3}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0. \tag{4}$$

If coefficient r does not depend on t and x, it is not difficult to study the stability of zero solution of the boundary value problem (3), (4) (i.e. the stability of the equilibrium $v_0 \equiv 1$ of the problem (1), (2)). In particular, it is well known that for $0 < r < \frac{\pi}{2}$ this solution is asymptotically stable and, if $r > \frac{\pi}{2}$, then it is unstable. If $r = \frac{\pi}{2}$, the nonlinear Eq. (1) with boundary conditions (2) undergoes the Andronov–Hopf bifurcation (see, e.g., [5]). Thus, the case when coefficient r is in certain sense close to $\frac{\pi}{2}$ is critical.

We study the stability of zero solution of the linear problem (3), (4), provided that the real coefficient $r = r(t,x)$ depends on t and x. In other words, we will be interested whether all solutions of this problem are bounded for $t \geq t_0$ or it has an unbounded solution on $[t_0, \infty)$. We answer these questions by constructing the asymptotic formulas for solutions of the boundary value problem (3), (4) as $t \to \infty$. This will be done in the situation when the coefficient $r = r(t,x)$ is close to $\frac{\pi}{2}$ in the following sense:

$$r(t,x) = \frac{\pi}{2} + r_1(t,x)t^{-\rho} + \cdots + r_k(t,x)t^{-k\rho} + O\big(t^{-(k+1)\rho}\big), \quad t \to \infty,$$

where $\rho > 0$. Here the real-valued functions $r_j(t,x)$ are the trigonometric polynomials in t and belong to the suitable functional space in x. The integer value $k \geq 0$ is choosen in the way that

$$(k+1)\rho > 1.$$

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ON BOUNDARY LAYER SOLUTIONS IN THE REACTION-DIFFUSION PROBLEM WITH A SINGULARLY PERTURBED NONLINEAR BOUNDARY CONDITION OF INTEGRAL TYPE

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Let us consider a singularly perturbed boundary value problem of reaction-diffusion type with a nonlinear singularly perturbed boundary condition of integral type:

$$\begin{aligned} &\varepsilon^2 \triangle u = f(u, x, \varepsilon), \quad x \in D, \\ &-\varepsilon \frac{\partial u}{\partial n}(x, \varepsilon) = \int\limits_{\Gamma} g(u(x,\varepsilon), u(s,\varepsilon), x, s)\, ds, \quad x \in \Gamma, \\ &-\frac{\partial u}{\partial n}(x, \varepsilon) = g^*(x), \quad x \in \Gamma^*, \end{aligned} \tag{1}$$

where $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2}$, D is a two-dimensional doubly connected domain bounded by sufficiently smooth simple closed curves Γ and Γ^* (Γ lies inside Γ^*), the derivative $\frac{\partial}{\partial n}$ is taken along the inner normal to the sufficiently smooth boundary $\partial D = \Gamma \cup \Gamma^*$, ε is a small parameter: $\varepsilon \in I_\varepsilon,\ I_\varepsilon := (0, \varepsilon_0]$, $0 < \varepsilon_0 << 1$.

The main result is a modification of Vasilyeva's boundary function method for constructing the asymptotics of solutions with a boundary layer, the development of an asymptotic method of differential inequalities for substantiating the existence of solutions uniformly close to the constructed asymptotics in the region under consideration, and a proof of the asymptotic stability in the sense of Lyapunov of such solutions as solutions of the corresponding initial-boundary value problem. These results are obtained for two cases: either the presence or absence of quasi-monotonicity in the integrand. Examples illustrating the results are given.

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ANALYTICAL THEORY FOR THE CONSTRUCTION OF WAVE AND LIGAMENT COMPONENTS OF SURFACE PERIODIC FLOWS

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Surface waves in liquids attract the attention of researchers due to the diversity of phenomena and the large number of both practical and academic applications. Gravitational waves exert destructive effects on bridges, dams, embankments, and other structures. Capillary phenomena occur at small scales — droplets are one of the causes of pathogenic bacteria propagation and are applied in industry — chemical, pharmaceutical, petroleum, and others. Observations and modern high–resolution experiments demonstrate the structured nature of periodic flows in continuous media [1]. Recent theoretical analytical studies [2–4] allow obtaining fine–structure components of flows. In constructing solutions, methods of singular perturbation theory are employed [5]. The mathematical formulation of the problem is based on the fundamental system of equations of fluid mechanics [6, 7] or on its reduced version. Solutions to the equations of motion are sought simultaneously for all physical quantities and with consideration of physically justified initial and boundary conditions. The high rank of the system of equations leads to the appearance of a large number of roots, describing flow components. The exact number of roots is determined by the order of the system of equations, which depends on the set of phenomena being considered. The present work is devoted to developing a methodology for calculating complete solutions to the equations of motion, incorporating both large-scale wave and fine-structure ligament components of periodic flows. The methodology is presented using the example of two-dimensional surface capillary-gravitational periodic flows in a viscous stratified fluid

To simplify the problem we neglect the non-uniform distribution of temperature and salinity of the fluid, i.e., the physical nature of stratification is not specified. Instead of an equation of state, an equilibrium density profile distribution along the vertical coordinate is prescribed, which for a wide range of problems can be considered exponential. Standard boundary conditions for this type of problem are prescribed on the free surface:

$$z<\zeta:\begin{cases}P=P_{00}+\int_z^\zeta g\rho\left(x,\xi,t\right)d\xi+\tilde{P}(x,z,t)\\ \rho\left(x,z,t\right)=\rho_{00}\left[r(z)+\tilde{\rho}(x,z,t)\right]=\rho_{00}\left[\exp\left(-z_\Lambda\right)+\tilde{\rho}(x,z,t)\right],\\ z_\Lambda=\dfrac{z}{\Lambda}\\ \rho\left(\partial_t\vec{u}+\left(\vec{u}\cdot\nabla\right)\vec{u}\right)=\rho\nu\Delta\vec{u}-\nabla P+\rho\vec{g}\\ \partial_t\rho+\vec{u}\cdot\nabla\rho=0\\ \operatorname{div}\vec{u}=0\end{cases}$$

$$z=\zeta:\begin{cases}\partial_t(z-\zeta)+\vec{u}\cdot\nabla(z-\zeta)=0\\ \vec{\tau}\cdot\left(\left(\vec{n}\cdot\nabla\right)\vec{u}\right)+\vec{n}\cdot\left(\left(\vec{\tau}\cdot\nabla\right)\vec{u}\right)=0\\ P-P_0-\sigma\operatorname{div}\vec{n}-2\rho\nu\vec{n}\left(\left(\vec{n}\cdot\nabla\right)\vec{u}\right)=0\end{cases}$$

Here P is the pressure in the fluid, P_{00} is the atmospheric pressure, $\tilde{P}(x,z,t)$ is the periodic pressure perturbation, ρ is the density, ρ_{00} is the equilibrium density value at the level $z=0$, $\tilde{\rho}$ is the small density perturbation associated with the propagation of the periodic surface flow $z=\zeta$, $r(z)=\exp\left(-z/\Lambda\right)$ is the function determining the equilibrium density distribution profile, $\Lambda=\left|d\ln\rho/dz\right|^{-1}$ is the stratification scale, which in an incompressible fluid is related to the buoyancy frequency $N=\sqrt{g/\Lambda}$, $\vec{u}$ is the fluid velocity, $\vec{n},\vec{\tau}$ are the normal and tangent vectors to the free surface, σ is the surface tension coefficient, ν is the kinematic viscosity.

The solution is sought in the form of harmonic functions of the type $f\left(x,z,t\right)\sim\exp\left(ik_xx-i\omega t\right)\exp\left(k_zz\right)$, in which the frequency $\omega>0$ is a real positive-definite quantity, and the components of the wave vector $\vec{k} = (k_x, k_z)$ may be complex and determine the spatial attenuation of flow components. The spectral form of the equations of motion is written as follows:

$$\omega\left(k_x^2 - k_z^2\right)\left(i\nu k_x^2 - i\nu k_z^2 + \omega\right) - N^2 k_x^2 \exp\left(-z_\Lambda\right) = 0$$

Using methods of singular perturbation theory, regular k_z and singular k_l solutions were obtained:

$$k_z = \pm\sqrt{k_x^2 - \frac{i\omega}{2\nu} - \frac{(1-i)\sqrt{4k_x^2\nu\omega N^2\exp\left(-z_\Lambda\right) - i\omega^4}}{2\sqrt{2}\nu\omega}},$$
$$k_l = \pm\sqrt{k_x^2 - \frac{i\omega}{2\nu} + \frac{(1-i)\sqrt{4k_x^2\nu\omega N^2\exp\left(-z_\Lambda\right) - i\omega^4}}{2\sqrt{2}\nu\omega}}.$$

Regular solutions correspond to large-scale wave components of flows. Singular solutions describe fine-structure ligament components. Taking into account the boundary conditions and complete solutions, a dispersion relation is written that determines the dependence of the wave number on the frequency of perturbations and other parameters of the problem:

$$\left(k_x^2 + k_z^2\right)\left(k_l\omega^2 - gk_x^2 - \gamma k_x^4 + i\omega\nu k_l\left(3k_x^2 - k_l^2\right)\right) -$$
$$- \left(k_x^2 + k_l^2\right)\left(k_z\omega^2 - gk_x^2 - \gamma k_x^4 + i\omega\nu k_z\left(3k_x^2 - k_z^2\right)\right) = 0.$$

The dispersion equation can be solved using numerical methods or using analytical asymptotic methods. The analytical solution is written as follows:

$$k_x = \frac{\alpha^{1/3}}{3^{2/3}2^{1/3}\gamma} - \frac{2^{1/3}g}{3^{1/3}\alpha^{1/3}},$$
$$\alpha = 9\gamma^2\omega\sqrt{\omega^2 - N^2} + \sqrt{3\gamma^3\left(4g^3 + 27\gamma\omega^2\left(\omega^2 - N^2\right)\right)}.$$

The proposed original methodology allows obtaining complete solutions to the equations of motion, as well as analyzing the dispersion characteristics of all flow components. The approach used makes it possible to identify not only wave components describing large-scale dynamics, but also ligaments that define the structure of flows. The application of the methodology can be extended and used not only for calculating surface periodic flows, but also for internal gravitational waves, acoustic waves, as well as hybrid oscillations.

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ON THE FEATURES OF THE SOLUTIONS WITH BOUNDARY LAYERS IN THE TIKHONOV SYSTEM OF ODEs WITH ADVECTION TERMS

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We consider a boundary value problem for a singularly perturbed system of fast and slow reaction-diffusion-advection equations. The problem is formulated as follows:

$$\begin{cases} \varepsilon^4 \dfrac{d^2u}{dx^2} - \varepsilon A(u,x)\dfrac{du}{dx} - g(u,v,x,\varepsilon) = 0, \quad 0 < x < 1; \\ \dfrac{d^2v}{dx^2} - B(v,x)\dfrac{dv}{dx} - f(u,v,x,\varepsilon) = 0, \end{cases}$$

where $\varepsilon \in (0, \varepsilon_0]$ is a small parameter.

For the component $u(x,\varepsilon)$, one of the following boundary conditions is imposed:

$$\frac{du}{dx}(0,\varepsilon) = u^0, \quad \frac{du}{dx}(1,\varepsilon) = u^1 \quad \text{or} \quad u(0,\varepsilon) = u^0, \quad u(1,\varepsilon) = u^1.$$

For the component $v(x,\varepsilon)$, Dirichlet conditions are set: $v(0,\varepsilon) = v^0$, $v(1,\varepsilon) = v^1$.

Boundary value problems for the studied system of ordinary differential equations (ODEs) arise in various applications, particularly in modeling fast bimolecular reactions [1] and in the study of polarization of eukaryotic cells in response to external stimuli during their division [2].

Using Vasilieva's method, we construct asymptotic expansions in the small parameter for boundary layer solutions under the specified boundary conditions. An essential feature of this problem is that the stretched variables used to determine boundary layer functions at the left and right endpoints depend on different powers of the small parameter. Using Nefedov's asymptotic method of differential inequalities [3], we prove existence and Lyapunov asymptotic stability theorems for solutions with the constructed asymptotics, both for quasimonotone sources and for sources without the quasimonotonicity requirement.

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TRAFFIC MODEL FOR TWO INTERACTING TRAFFIC STREAMS

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This paper presents a novel microscopic traffic flow model that captures multi-lane traffic dynamics while incorporating lane-changing behavior for two interacting traffic streams and the formation of bottleneck situations.

Let the superscript y denote all parameters and functions related to the vehicle from flow Y, and the superscript x represent everything associated with the vehicles from flow X.

Assume that a vehicle from flow Y can merge into the moving flow X only if the distance between it and the vehicle it intends to follow, as well as the distance between it and the vehicle in front of which it merges, exceeds the stopping distance. In this case, three distinct scenarios may arise:

1. The vehicle from flow Y becomes the first in flow X.
2. The vehicle from flow Y merges into the middle of flow X.
3. The vehicle from flow Y becomes the last in flow X.

The first scenario describes the situation where a vehicle from flow Y becomes the first in stream X. To determine whether this merging is successful, a relay function of the following form is introduced:

$$W_1 = \begin{cases} 1, & \text{if } \Delta y_1 > S_1^x, \\ 0, & \text{if } \Delta y_1 \le S_1^x, \end{cases} \qquad S_1^x = (\tau_1^x + \tau_{b,1}^x)\dot{x}_1(t) + \frac{\dot{x}_1^2(t)}{2\mu g} + l_1^x.$$

If the relay function W_1 allows the vehicle from flow Y to become the first in flow X, then it must continue moving as the lead vehicle in flow X:

$$\ddot{y}(t) = R_1^y \left[a^y \left(v_{max}^y - \dot{y}(t)\right)\right] - (1 - R_1^y)H_1^y.$$

The second scenario describes the situation where a vehicle from flow Y merges between the $n-1$ and n vehicles in stream X. For a successful lane change, the distances to both the vehicle ahead (the $n-1$ vehicle) and the vehicle behind (the n vehicle) must exceed the required braking distance. To determine whether this maneuver is possible, the following relay function is introduced:

$$W_n = \begin{cases} 1, & \text{if } \Delta' y_{n-1} > S^y \text{ and } \Delta y_n > S_n^x, \\ 0, & \text{if } \Delta' y_{n-1} \leq S^y \text{ or } \Delta y_n \leq S_n^x, \end{cases}$$

$$S_n^x = (\tau_n^x + \tau_{b,n}^x)\dot{x}_n(t) + \frac{\dot{x}_n^2(t)}{2\mu g} + l_n^x.$$

If the relay function W_n permits the vehicle from stream Y to merge between the $n-1$ and n vehicles in stream X, the newly merged vehicle must now adjust its behavior based on the movement of the $n-1$ vehicle from stream X:

$$\ddot{y}(t) = R_n^y \left[a^y \left(P_n^y - \dot{y}(t)\right)\right] - (1 - R_n^y) H_n^y.$$

The third scenario describes a situation where a vehicle from flow Y becomes the last in stream X. For this lane change to occur, the only requirement is that the distance to the last vehicle in flow X exceeds the stopping distance of the merging vehicle.

$$W_N = \begin{cases} 1, & \text{if } \Delta' y_N > S^y, \\ 0, & \text{if } \Delta' y_N \leq S^y, \end{cases} \qquad S^y = (\tau^y + \tau_b^y)\dot{y}(t) + \frac{\dot{y}^2(t)}{2\mu g} + l^y.$$

If the relay function W_N allows the vehicle from flow Y to become the last in flow X, from that point onward, it should adjust its behavior based on the actions of the last (N-th) vehicle in flow X:

$$\ddot{y}(t) = R_N^y \left[a^y \left(P_N^y - \dot{y}(t)\right)\right] - (1 - R_N^y) H_N^y.$$

Before the lane change, the vehicle follows the dynamics of its original flow Y, assuming it is the only vehicle in that flow:

$$\ddot{y}(t) = R^y \left[a^y \left(v_{max}^y - \dot{y}(t)\right)\right] - (1 - R^y) H^y.$$

Thus, the movement of a vehicle from stream Y can be described as the product of a vector of relay functions for lane changes and a vector of possible driving scenarios. Since the relay function vector has only one non-zero element at any given time t, while the other elements are zero, this product uniquely determines the driving scenario for the vehicle from stream Y.

$$\begin{cases} \ddot{y}(t) = \begin{pmatrix} W_1 \\ \dots \\ W_n \\ \dots \\ W_N \\ \prod_{i=1}^{N}(1-W_i) \end{pmatrix} \begin{pmatrix} R_1^y \left[a^y \left(v_{max}^y - \dot{y}(t)\right)\right] - (1-R_1^y)H_1^y \\ \dots \\ R_n^y \left[a^y \left(P_n^y - \dot{y}(t)\right)\right] - (1-R_n^y)H_n^y \\ \dots \\ R_N^y \left[a^y \left(P_N^y - \dot{y}(t)\right)\right] - (1-R_N^y)H_N^y \\ R^y \left[a^y \left(v_{max}^y - \dot{y}(t)\right)\right] - (1-R^y)H^y \end{pmatrix} \\ y(t) = \lambda^y, \quad \dot{y}(t) = v^y, \quad \text{for } t \in [-\tau^y, 0]. \end{cases} \tag{1}$$

In lane X, if a vehicle from lane Y merges, it influences the vehicle ahead as part of flow X. Without a merge ahead, vehicles follow their lane leader as before; if the Y-vehicle becomes last, it has no effect on others.

For the lead vehicle in flow X, this can be expressed as follows:

$$M_1 = \begin{cases} \bar{R}_1^x \left[a_1^x \left(\bar{P}_1^x - \dot{x}_1(t)\right)\right] - (1-\bar{R}_1^x)\bar{H}_1^x, & \text{if } y(t-\tau_1^x) > x_1(t), \\ R_1^x \left[a_1^x \left(v_{max,1}^x - \dot{x}_1(t)\right)\right] - (1-R_1^x)H_1^x, & \text{if } y(t-\tau_1^x) \le x_1(t). \end{cases}$$

For the remaining vehicles in the flow, the system can be expressed as follows:

$$M_n = \begin{cases} \bar{R}_n^x \left[a_n^x \left(\bar{P}_n^x - \dot{x}_n(t)\right)\right] - (1-\bar{R}_n^x)\bar{H}_n^x, & \text{if } y(t-\tau_n^x) > x_n(t) \text{ and} \\ & \quad y(t-\tau_{n-1}^x) < x_{n-1}(t), \\ R_n^x \left[a_n^x \left(P_n^x - \dot{x}_n(t)\right)\right] - (1-R_n^x)H_n^x, & \text{if } y(t-\tau_n^x) \le x_n(t) \text{ or} \\ & \quad y(t-\tau_{n-1}^x) \le x_{n-1}(t). \end{cases}$$

The system of equations describing the movement of vehicles in lane X, accounting for the merging vehicle from lane Y, is given by:

$$\begin{cases} \ddot{x}_1(t) = M_1, \\ \ddot{x}_n(t) = M_n, \\ \quad x_n(t) = \lambda_n^x, \quad \dot{x}_n(t) = v_n^x, \quad \text{for } t \in [-\tau_n^x, 0]. \end{cases} \tag{2}$$

Thus, the systems of equations (1) and (2) comprehensively describe the interaction between two traffic flows, accounting for all possible scenarios where the second flow consists of a single vehicle.

LARGE N LIMIT OF CLASSICAL GAUDIN MODEL AND SPECTRAL DUALITY

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The rational $\mathfrak{gl}_N$ classical Gaudin model with M poles is defined on the product $\mathfrak{gl}_N^{*\times M}$ with Lie-Poisson bracket by Lax matrix with simple poles

$$L(z) = V + \sum_{k=1}^{M} \frac{S^k}{z - z_k} \in \mathrm{Mat}_N \,, \tag{1}$$

and constant diagonal twist matrix $V = \mathrm{diag}(v_1, ..., v_N)$. For rank $= 1$ variables $S^k_{ij} = \xi^k_i \eta^k_j \in \mathrm{Mat}_N$ one may construct Gaudin model on $\mathfrak{gl}_M^{*\times N}$ with dynamical residues $\tilde{S}^k_{ab} = \xi^a_k \eta^b_k \in \mathrm{Mat}_M$

$$\tilde{L}(v) = Z + \sum_{k=1}^{N} \frac{\tilde{S}^k}{v - v_k} \in \mathrm{Mat}_M \,, \tag{2}$$

and twist matrix $Z = \mathrm{diag}(z_1, ..., z_M)$, such that their spectral curves coincide [1]:

$$\det(z - Z)\det(v - L(z)) = \det(v - V)\det(z - \tilde{L}(v)) \,. \tag{3}$$

This correspondence is the example of spectral duality of classical integrable models.

Infinite-dimensional and field-theoretical Gaudin models arise from $N \to \infty$ limit of Mat_N associative algebra, based on the representation of noncommutative torus algebra (NCT) by infinite matrices with coefficients in $\mathbb{C}[[\hbar^{-1}, \hbar]]$ [2]. Integrable systems of such type are described by Lax operators with values in NCT. In this talk I will describe the $N \to \infty$ limit of the rational Gaudin model (1) and show how spectral duality may be generalized to field-theoretical case.

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The talk is based on our upcoming work with Andrei Zotov.

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HOMOTHETY OF PERIODIC SOLUTIONS IN A CLASS OF DELAY DIFFERENTIAL EQUATIONS

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Let us consider a class of differential equation with delays, written in general form:

$$F(x, x^{(1)}, \dots, x^{(n)}, x(t-\tau_1), \dots, x(t-\tau_m)) = 0. \tag{1}$$

Here $x = x(t)$ is a scalar function, $x^{(i)} = x^{(i)}(t)$ denotes the i-th derivative, time delays τ_i, $i = 1, \dots, m$, are positive, $F : \mathbb{R}^{n+m+1} \to \mathbb{R}$. Let $x^*(t)$ be a known T^*-periodic solution of equation (1). We pose the question of the existence and stability of a periodic solution of the form

$$y^*(t) = \alpha x^*(\beta t), \tag{2}$$

where α and β are some positive fixed constants. Note that in the case $\alpha\beta = 1$ the map $x^*(t) \mapsto y^*(t)$ geometrically means a homothety with coefficient β.

The function $y^*(t)$ described by formula (2) is a solution of (1) if the function F and the delays τ_i, $i = 1, \dots, m$, satisfy the following property.

Proposition 1. *Let scalar functions $x_i = x_i(t)$, $i = 0, 1, \dots, n$, $y_j = y_j(t)$, $j = 1, \dots, m$, be such that*

$$F(x_0, x_1, \dots, x_n, y_1, \dots, y_m) = 0. \tag{3}$$

There exist $\alpha, \beta > 0$ such that (3) implies the equality

$$F(\alpha x_0, \alpha\beta x_1, \dots, \alpha\beta^n x_n, \alpha y_1, \dots, \alpha y_m) = 0. \tag{4}$$

In the case $\alpha\beta = 1$ (4) takes the form

$$F(x_0/\beta, x_1, \beta x_2, \ldots, \beta^{n-1}x_n, y_1/\beta, \ldots, y_m/\beta) = 0.$$

Proposition 2. *For any $i, j \in \{1, \ldots, m\}$ $\tau_i/\tau_j \in \mathbb{Q}_+$.*

Let us look at some examples.

Example 1. *In [1], for $n = m = 1$, $\tau_1 = 1$, $F(x_0, x_1, y_1) = x_1 - \operatorname{sign}(y_1)$, it has been proven that equation (1) has a countable family of solutions $y_k^*(t) = \beta_k x^*(t/\beta_k)$, $k = 0, 1, \ldots$, where $\beta_k = 1+4k$,*

$$x^*(t) = \begin{cases} t & t \in [0,1], \\ -t+2 & t \in [1,3], \\ t-4 & t \in [3,4], \end{cases} \quad x^*(t+4) \equiv x^*(t).$$

Moreover, the solution is stable only for $k = 0$ and is not stable for $k \in \mathbb{N}$. Here $F(x_0/\beta, x_1, y_1/\beta) = F(x_0, x_1, y_1)$ for any positive β.

Example 2. *In [2], for $n = 1$, $m = 2$, $\tau_1 = 1$, $\tau_2 = h$, $F(x_0, x_1, y_1, y_2) = x_1 - 1 + (a+1)H(y_1) - bH(y_2)$, $a, b > 0$, $h \in \mathbb{Q}_+$, H is Heaviside function, it has been proven that equation (1) has also a countable family of solutions $y_k^*(t) = \beta_k x^*(t/\beta_k)$, $k = 0, 1, \ldots$, where $\beta_k = 1+kT^*$, $x^*(t)$ is a T^*-periodic solution (1) in this case constructed in work [3]. The solution $y_k^*(t)$ is stable [3] only for $k = 0$ and is not stable [2] for $k \in \mathbb{N}$. Here $F(x_0/\beta, x_1, y_1/\beta, y_2/\beta) = F(x_0, x_1, y_1, y_2)$ for any positive β.*

Example 3. *In [4], for $n = m = 2$, $\tau_1 = 1$, $\tau_2 = N$, $F(x_0, x_1, x_2, y_1, y_2) = x_2 - H(y_1) - H(y_2) + 2H(x_0)$, $N \in \mathbb{N}$, H is Heaviside function, it has been proven that equation (1) has an unstable solution $y^*(t) = Nx^*(t/N^2)$, where $x^*(t)$ is an unstable periodic solution (1) in this case constructed in work [4]. Here $F(x_0/\beta, x_1, \beta x_2, y_1/\beta, y_2/\beta) = F(x_0, x_1, x_2, y_1, y_2)$ for any positive β, the map $x^*(t) \mapsto y^*(t)$ is not a homothety.*

The objective of this paper is to describe a suitable class of functions F, find other meaningful examples of equation (1) that have solutions in form of (2) (in particular homothetic solutions), and investigate the stability of these solutions.

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RESEARCH OF SELF-OSCILLATORY SOLUTIONS OF AN INITIAL BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL FUNCTIONAL EQUATION ARISING IN THE MECHANICS OF ROTARY SYSTEMS

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We investigate self-oscillatory solutions of the initial-boundary value problem for the functional-differential equation

$$
\begin{gathered}
u_{tt} + (a_0 + a(|u_t|^2))u_t + (\mathrm{I} - \mu\Gamma(i\Omega))(u_{xx} + b(|u_{xx}|^2)u_{xx})_{xx} = 0, \\
(\mathrm{I} - \mu\Gamma(i\Omega))\varepsilon(t) \equiv \varepsilon(t) - \mu \int\limits_{-\infty}^{0} R(\tau)\mathrm{e}^{-i\Omega\tau}\varepsilon(t+\tau)d\tau, \\
\int\limits_{-\infty}^{0} R(\tau)d\tau = 1,\ 0 < \mu < 1
\end{gathered}
\tag{1}
$$

regarding the function $u(x, t+\tau) = y(x, t+\tau) + iz(x, t+\tau)$, $(i = \sqrt{-1})$ in the area of $\bar{Q}_+ = \{0 \le x \le 1,\ 0 \le t < \infty\}$ A three-point initial boundary value problem with boundary conditions is considered

$$
\begin{gathered}
u|_{x=0} = u_x|_{x=0} = 0,\ u|_{x=1} = u_x|_{x=1} = 0, \\
u|_{x=l-0} = u|_{x=l+0},\ u_x|_{x=l-0} = u_x|_{x=l+0},
\end{gathered}
$$

$$(\mathrm{I} - \mu\Gamma(i\Omega))(u_{xx} + b(|u_{xx}|^2)u_{xx})|_{x=l+0}^{x=l-0} = \\ = -Ju_{ttx}|_{x=l} - G_1(a_0 + a(|u_{xt}|^2))u_{xt}|_{x=l},$$

$$(\mathrm{I} - \mu\Gamma(i\Omega))(u_{xx} + b(|u_{xx}|^2)u_{xx})_x|_{x=l+0}^{x=l-0} = \\ = Mu_{tt}|_{x=l} + G_2(a_0 + a(|u_t|^2))u_t|_{x=l},$$

$0 < l < 1$ and the initial conditions

$$u(x, t+\tau)|_{t=0} = g_0(x,\tau) \in D(\bar{Q}_-),\ u_t(x,0) = g_1(x) \in H^2, \quad (2)$$

In formulas (1)-(2) $R(\tau) > 0$, $\infty < \tau < 0$, there is a continuous convex downward function (relaxation function), the functions $a(\xi) = a_1\xi + ...$ and $b(\xi) = b_1\xi + ...$ are assumed to be infinitely differentiable, $a_0, \Omega, M, J, G_1, G_2, a_1, b_1$ are positive parameters, the expression $u|_{x=l+0}^{x=l-0} = u|_{x=l-0} - u|_{x=l+0}$, and $D(\bar{Q}_-)$, $\bar{Q}_- = \{0 \le x \le 1, -\infty < \tau \le 0\}$ and H^2 are the function spaces of the initial conditions. The initial boundary value problem (1)-(2) is a mathematical model of the dynamics of a rotating ideal flexible shaft of constant circular cross-section made of a material with nonlinear- the resulting properties [1], the ends of which are supported by bearings, with an ideal solid circular disk mounted on the shaft. In (1)–(2), the function $a_0 + a(\xi)$ characterizes the coefficient of external viscous friction, the function $b(\xi)$ is determined by a nonlinear function of deformation of the shaft material, M and J respectively characterize the mass of the disk and its moment of inertia relative to an arbitrary axis perpendicular to the midline of the shaft, the dimensionless coefficients G_1 and G_2 characterizes the geometric properties of the disk, the parameter Ω is the velocity of rotation [2]. The conditions of stability of the horizontal shape of rotation, mecha are investigated. The loss of stability is low, with bifurcating self-oscillatory movements, their nature, stability, and dependence on the location of the disk.

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CCHAOS NEAR-RESONANCE IN PERTURBED COUPLED NONLINEAR SCHRÖDINGER EUQTIONS

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We consider a system of coupled nonlinear Schrödinger equations with even, periodic boundary conditions, which are damped and quasi-periodically forced. Under certain conditions, we prove the existence of manifolds of multi-pulse Šilnikov-type orbits homoclinic to a spatially independent invariant torus. Moreover, chaos is induced under generic conditions. Using the method or time-delayed embedding, a signal can be embedded into higher-dimensional space in order to study its dynamics. In a dynamical system, the Lyapunov exponents measure the exponential divergence of initially close trajectories, with positive Lyapunov exponents indicating divergent trajectories and negative Lyapunov exponents indicating convergence. A chaotic system, due to its sensitivity to initial conditions, must have at least one positive Lyapunov exponent. Furthermore, an increasingly chaotic system has an increasing number of positive Lyapunov exponents. This section evaluates Lyapunov exponents of the coupled perturbed NLS for increasing domain size L and interprets each increase in the number of positive Lyapunov exponents as a transition to an increasingly chaotic system.

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ON ADJACENCIES OF SIMPLE STABLE SINGULARITIES OF A FRONT

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The image of a Legendrian map is called a front. The equidistant of a smooth cooriented hypersurface is an example of a front. A germ of any front is a germ of the bifurcation diagram of zeros for a family of smooth functions. By Arnold’s theorem on Legendrian singularities, simple stable Legendrian germs are defined by versal deformations of germs of functions at critical points of types A, D, E. We will talk about the calculation of the adjacency indices of type D singularities of a front.

BASINS OF WADA WITH A DIFFERENT ROTATION NUMBERS

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In [1] authors have researched the geometric anatomy of the non-wandering continua possessing the Wada property turning out to be limits of inverse spectrums of current lines defining by the simplest natural model delivering a multi-valued map. In order to we chosen the equation for n vortex filaments parallel to each other, equidistant and located on a circular cylinder of radius R, assuming the intensity Γ of the vortices to be the same. It is clear that the rotation numbers for all basins turn out to be the same.

Now let us create the situations with two Wada basins with different rotation numbers.The hydrodynamic approach also proves fruitful in this case. Indeed, the equation for two vortices with intensities $\Gamma_1 \neq \Gamma_2$

$$v_x - v_y = \frac{1}{2\pi i}\left(\frac{\Gamma_1}{z-R} + \frac{\Gamma_2}{z+R}\right)$$

delivers the current lines (set $\mathcal{A}_0$) in polar coordinates ($z \stackrel{\text{def}}{=} \varrho \cdot \exp^{i\varphi}$) as follows

$$(\varrho^2 - 2\varrho R\cos\varphi + R^2)^{\Gamma_1}(\varrho^2 + 2\varrho R\cos\varphi + R^2)^{\Gamma_2} = \\ = (C + R\tanh\cot\varphi)^{2(\Gamma_1+\Gamma_2)},$$

where

$$R \cdot \frac{\Gamma_1 - \Gamma_2}{\Gamma_1 + \Gamma_2} \leq C \leq \frac{2R}{3}.$$

Therefore the rotation numbers ratio $\alpha_1 : \alpha_2$ for the Wada basins is determined by the intensities ratio $\Gamma_1 : \Gamma_2$ (on this occasion, see also [2]).

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NUMERICAL AND ASYMPTOTIC ANALYSIS OF THE SOLUTION OF A MODEL PROBLEM ON THE SPREAD OF TUMOR CELLS

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A system of equations is considered

$$\frac{\partial u}{\partial t} - \varepsilon^2 D_u \Delta u = -\begin{cases} u, & [H^+] \geq [H^+]_{thr}, \\ \begin{cases} (u - u_{max}), & u \geq u_{thr}, \\ -k_1 \cdot u \cdot [O_2] & u < u_{thr}, \end{cases} & [H^+] < [H^+]_{thr}; \end{cases}$$

$$\frac{\partial [H^+]}{\partial t} - \varepsilon^2 D_{[H^+]} \Delta [H^+] = -\begin{cases} ([H^+] - [H^+]_{max}), & [H^+] \geq [H^+]_{thr}, \\ -k_2 \cdot u, & [H^+] < [H^+]_{thr}. \end{cases}$$

This system represents a reduced version of the tumor spheroid spread model [1]. Here u is the tumor cell density normalized to 1 (thus $u_{max} = 1$), $[H^+]$ is the concentration of cations, $[O_2]$ is oxygen concentration, k_1 is cell division rate, k_2 is glycolysis constant, $[H^+]_{thr}$ is the threshold concentration of cations above which the environment becomes too acidic for cell life, u_{thr} is the threshold value of normalized cell density below which proliferation occurs most intensively.

Sufficient oxygen concentration is necessary for tumor cell proliferation (division), while with an increase in their number, the cells inside the tumor experience hypoxia, under which glycolysis in cells (oxidation of glucose contained in the cell) occurs with the release of cations, $[H^+]$. As a result of the latter, the pH shifts to the acidic side, ultimately causing necrosis of tumor cells in its interior. Thus, the distribution of tumor cells in the one-dimensional case has the appearance of two diverging humps.

Theorems on the existence of solutions for systems of this type from the space $\mathbb{L}_2$ are given in the work [2].

The numerical solution in the form of a two-humped distribution was carried out using the CROS2 method with Richardson accuracy control.

For the given one-dimensional problem, a study was conducted using the asymptotic analysis method. An analytical dependence of the tumor propagation rate on the parameters k_1, k_2, u_{thr}, $[H^+]_{thr}$ and on the oxygen concentration was obtained.

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ON THE QUANTUM ARGUMENT SHIFT METHOD

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In my previous talks on this topic, I described an axiomatic way to define the operator of "quantum shift" on the algebra Ugl_d. This time I will try to take another point of view and give a more geometric description of this procedure (and try to prove that it works).

DISPERSIVE SHOCK WAVES FOR INTEGRABLE EQUATIONS IN AKNS SCHEME WITH DISSIPATIVE PERTURBATION TERMS

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We consider integrable equations in the framework of the AKNS scheme with dissipative perturbation terms. After breaking of the initial step, ordinary dispersive shock waves are stabilized [1] due to dissipation and move with stationary profiles. We present here a technique to describe such a problem in the framework of the Gurevich-Pitaevskii approach using the perturbation Whitham theory [2]. This method allows us to obtain several conservation laws linking the structure parameters at the edges, as an analogue of the Renkine-Hugoniot relations, and to fully describe the wave profile. We demonstrate how it works on example of the Kaup-Boussinesq-Burgers equation [3] and nonlinear Schrödinger equation with Raman term [4].

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REAL SOLUTIONS TO HITCHIN SYSTEMS

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We obtain local real solutions to Hitchin systems with simple structure groups by using a Jacoby-type reversion on real parts of invariant tori of those systems.

ANALYZING PHASE SPACE TRANSPORT USING THE ORIGIN FATE MAP

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Following a recent publication [1], we introduce and apply the origin-fate map (OFM) as a powerful tool for the detailed analysis of phase space transport in reactant–product–type systems. These systems, characterized by well-defined initial (reactant) and final (product) states, allow for a comprehensive understanding of the underlying manifold dynamics through the backward and forward integration of initial conditions, which are then indexed according to their origin and fate. We demonstrate the method using a two degrees of freedom caldera potential with four exit channels. Our results show that the OFM not only reproduces key features captured by classical manifold theory but also reveals finer details, such as intricate lobe structures and dynamically significant transitions arising from lobe formation. Overall, the OFM is a conceptually simple yet highly effective tool, offering broad applicability for both qualitative visualization and quantitative analysis of complex dynamical systems.

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NON-LINEAR SUPERSYMMETRY FOR NON-HERMITIAN MATRIX HAMILTONIANS AND ITS MINIMAL REALIZATIONS

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We consider different types of minimization of a matrix intertwining operator, criterions for these types of minimization and their proofs.

ON THE LINK BETWEEN OSCILLATION OF THE SECOND ORDER ODE AND THE FIRST ORDER DELAY EQUATION

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We consider the oscillation theory of the second order differential equation

$$x''(t) + p(t)x(t) = 0, \tag{1}$$

where

$$p(t) \geq 0,$$

starting with the classical results (e.g. [1]).

The oscillation of (1) is known to be equivalent to the oscillation of the first order delay equation

$$y'(t) + (\frac{1}{e} + \frac{p(t)}{2e})y(t-1) = 0, \tag{2}$$

which is at the critical state [5].

A recent, independent, center-manifold approach constructed a two-dimensional ODE system, whose oscillation is also equivalent to the oscillation of (2), even when $p(t)$ is allowed to take both positive and negative values [4]. What is more, we prove that the two-dimensional center manifold system coincides with (1), up to some error terms [3].

We finally take a retrospective look at certain methods of constructing characteristic equations for nonautonomous equations [2], which were instrumental in the first results connecting the oscillation of (1) and (2).

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THE LIE-BIANCHI INTEGRABILITY OF THE FULL SYMMETRIC TODA SYSTEM

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This talk is devoted to the Lie-Bianchi integrability of the full symmetric Toda system, i.e. to the fact that there exists a solvable Lie algebra of vector fields of dimension on the phase space of this system such that the system is invariant with respect to the action of these fields. The proof is based on the use of symmetries of the full symmetric system, which we described earlier in [1], and the appearance of the structure of the stochastic Lie algebra in their description. The talk is based on the joint work with Yu. Chernyakov and G. Sharygin arXiv:2506.07113.

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CELLULAR STRUCTURES AS THE BASIS OF LOGICAL UNITY OF CLASSIC AND QUANTUM THEORY

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Cellular structures have attracted attention in mathematical physics in 2000s due to newly discovered "isospectrality / non-isometry" effect (notation "$\sim$ / $\not\approx$"). Namely, there was found a whole gallery [1] of examples of non-isometric cellular domains $\Omega \not\approx \widetilde{\Omega}$ with the same energy spectra $E_n = \widetilde{E}_n, n = 1, 2, \ldots$, which are computed from Schroedinger's equation $-\Delta\Psi_n = E_n\Psi_n, \Psi_n\mid_{\partial\Omega}= 0$. Such cellular domains are very partial case of well-known cellular spaces [2] and may be restricted to CW-complexes. These examples imply negative answer to the classic acoustic version of this problem "Can one hear the shape of a drum?" (M. Kac, 1966), where Ω = drum's membrane, Ψ_n = amplitudes, E_n = frequencies of its eigenvibrations.

Definition. Cell is a connected subdomain $\Phi \subset \Omega : Spec(\Phi) \subset Spec(\Omega)$ with its wavefunction base $\{\psi_k\}_{k=1}^{\infty}$ to be the restriction of a global subset $\{\Psi_{n_k}\}_{k=1}^{\infty}$:

$$\psi_k = \Psi_{n_k}\mid_{\Phi}, \Psi_{n_k}\mid_{\partial\Phi}= 0 \Leftrightarrow e_k = E_{n_k}, e_k \in Spec(\Phi), E_{n_k} \in Spec(\Omega) \quad (1)$$

(for example any Ω with a symmetry axis has 2-cellular structure, Ω which is similar to a chess board has 64-cellular structure). Cellular domain Ω is constructed by means of mirror reflections of its cell Φ over ribs (= line segments) on $\partial\Phi$ and cell's wavefunctions ψ_k are extended acc. to (1) to the wavefunctions Ψ_{n_k} by odd extensions.

At the ISND-2024 the author firstly presented a new general method revealing the key role of cellular domains in the "isospectrality / non-isometry" effect and explaining the examples in [1]. Isospectrality conditions $E_n = \widetilde{E}_n, \forall n$, give equalities of Sobolev norms of the corresponding wavefunctions which are orthonormed in L_2: $\|\Psi_n\|^2_{\overset{\circ}{W}{}^1_2(\Omega)} = 1 + E_n = 1 + \widetilde{E}_n = \left\|\widetilde{\Psi}_n\right\|^2_{\overset{\circ}{W}{}^1_2(\widetilde{\Omega})}, \forall n$. Then $T : \widetilde{W_n} = TW_n, \forall n$, is a double unitary operator preserving Lebesgue and Sobolev norms, and this is the first step of the new method which gives the following result (see complete proofs in [3]):

Theorem 1. *If Ω and $\widetilde{\Omega}$ are isospectral $\Omega \sim \widetilde{\Omega}$ and non-isometric $\Omega \not\approx \widetilde{\Omega}$, then they are m-cellular domains with the same cell Φ, they have multi-valued isometry $\mathbf{T}\Omega = \widetilde{\Omega}$ and any cellular subdomain $\omega \subset \Omega$ has isospectral cellular image $\mathbf{T}\omega \subset \widetilde{\Omega}$:*

$$\omega = \mathbf{T}^{-1}\mathbf{T}\omega \Leftrightarrow Spec(\omega) = Spec(\mathbf{T}\omega). \tag{2}$$

Here for each pair of cells $\Phi_i \subset \Omega$ and $\widetilde{\Phi}_j \subset \widetilde{\Omega}$ the restriction of multi-valued isometry $\mathbf{T}\Phi_i \mid_{\widetilde{\Phi}_j}$ is usual point-wise isometry. Note two consequences of principal importance:

Corollary 1. *There exist merely two realizations of isospectrality: $\mathbf{T}$ = isometry and $\mathbf{T}$ = multi-valued (cellular) isometry.*

Corollary 2. *Acc. to (2) there inevitably exist disconnected isospectral cellular subdomains due to the cellular structure of global domains Ω and $\widetilde{\Omega}$.*

These results have a rather unexpected application in algebraic logic [5], [6]. Namely the pure geometric condition of the cellularity on the left of (2) implies that we can extend isospectrality from cellular subdomains to arbitrary cellular subsets $\omega \subset \Omega$. Then any intersections, unions, complements and closures of these "isospectral" subsets are themselves "isospectral" and form an algebra $\mathbf{S}$ of subsets. Introducing a topology acc. to the rule that a set $\omega \subset \Omega$ is open if it is isospectral to its image $\mathbf{T}\omega \subset \widetilde{\Omega}$ we have the well-known **extremally disconnected** topology [5] and $\mathbf{S}$ is the algebra of all its open-closed subsets (Stone's cellular algebra). The key importance of such algebras in algebraic logic may be expressed combining Lindenbaum-Tarski and Stone theorems [4], [5]:

Theorem 2 (Lindenbaum-Tarski-Stone). *Any logical theory i.e. consistent complete propositional calculus is isomorphic to Stone's cellular algebra* $\mathbf{S}$.

There already exists well-known interpretation of the quantum theory as a logical theory developed by G. Birkhoff and J. von Neumann [6]. Within this interpretation the distributive property of classic logic was modified to suit to quantum effects. On the contrary the direct transition from Schroedinger's equation to the cellular algebra does not require any axiomatic changes but reveals the logic that already lies in the foundations of the quantum theory. Using isospectral / non-isometry examples of Ω and $\widetilde{\Omega}$ from [1], the geometric condition of the cellularity (2) permits to draw visual models of the main logic effects including transition of quantity into quality, wave-particle duality etc. It reveals a generalization of the double negation law for representation of any statement $\omega \subset \Omega$:

$$\omega \subseteq \mathbf{T}^{-1}\mathbf{T}\omega, \tag{3}$$

where the equality "=" corresponds to the law of excluded middle. The formula (3) is well-known in mathematical logic [7] as an axiom but here we strictly derive it from Th. 2 and properties of Schroedinger's isospectral sets. This approach permits to visualize "invisible" logical relations similar to Wilson cloud chamber which permits to see geometric tracks of invisible particles.

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DISCRETE TRAVELING WAVES IN A TODA RELAY CHAIN: NEW RESULTS

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We consider a system of nonlinear equations describing the dynamics of interconnected nonlinear oscillators. The general form of the system is [1]:

$$M\ddot{r}_j = f(r_{j+1}) + f(r_{j-1}) - 2f(r_j).$$

Here, $r_j(t)$ denotes the displacement of the j-th spring from its equilibrium length, and the function f characterizes the spring tension force.

This type of system was first studied by Morikazu Toda in the context of investigating the dynamic behavior of crystal lattice elements. In his works [2,3], he examined solutions of the equation for $f(r) = -\alpha(1 - e^{-\beta r})$.

In the paper [4], a similar problem was considered for a ring of oscillators, where the Heaviside step function was chosen for f. In this talk, we will discuss new solutions for the system under consideration, as well as approaches to finding them and possible generalizations.

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MOVING FRONTS IN SINGULARLY PERTURBED REACTION-ADVECTION-DIFFUSION PROBLEMS

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This review synthesizes recent research on the dynamics of moving fronts within singularly perturbed reaction-advection-diffusion systems. These systems are characterized by a small parameter ε, which significantly influences to the structure of the solution, leading to the formation of sharp internal transition layers or moving fronts.

The main focus is on describing the movement and stabilization of fronts under various reaction, diffusion, and advection conditions. Applications of asymptotic methods for the development of approximations for solutions with moving fronts, in particular the Vasil'eva boundary function method, are considered. It also discusses the use of Nefedov's asymptotic method of differential inequalities to establish theorems on the existence and uniqueness of these solutions, accommodating both to the cases of continuous and discontinuous nonlinearities.

Significant attention is devoted to the consideration of the influence of nonlinear diffusion on the dynamics of the front. Applications of this research include modeling in nonlinear heat conduction, where intense sources and variable heat conductivity are considered, and in population dynamics, where rapid multiplication and diffusion dependent population growth are the key factors.

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MULTISTABLE PERIODIC SOLUTIONS IN NEUTRAL DELAY DIFFERENTIAL EQUATION INDUCED BY AN INFINITE DIMENSIONAL BIFURCATION

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Consider a differential equation of the form

$$\dot{x} + x = p \cdot \dot{x}(t-T) + F(x, x(t-T), \dot{x}(t-T)), \tag{1}$$

where $p \in \mathbb{R}$, $T \in \mathbb{R}_{>0}$ is a positive constant delay, and $F(x,y,z) = o(x,y,z)$ is a non-linearity.

This equation arises as a model in control systems, where nonlinear control force depends primarily on a velocity of the control variable.

The stability of zero equilibrium solution [1] is studied from the linearised equation

$$\dot{x} + x = p \cdot \dot{x}(t-T). \tag{2}$$

Theorem 1. *The roots to the characteristic equation*

$$\lambda + 1 = p \cdot \lambda e^{-\lambda T} \tag{3}$$

of linear equation (2):

- *all have negative real part if* $|p| \leqslant 1$*;*
- *all but a finite number have positive real part if* $|p| > 1$.

This constitutes a rather unique case when simultaniously infinite amount of characteristic roots cross imaginary axis with a small change of parameter.

Theorem 2. *The roots to equation (3) can be enumerated as* λ_n *(*$n \in \mathbb{Z}_{\geqslant 0}$*) with even* n *for* $p > 0$ *and odd* n *for* $p < 0$*, such that the following asymptotic formula holds:*

$$\lambda_n = i\frac{\pi n}{T} + \frac{\ln|p|}{T} + \frac{i}{\pi n} - \frac{T + 2\ln|p|}{2\pi^2 n^2} + O\left(\frac{1}{n^3}\right). \qquad (4)$$

We focus our attention on a particular case $F(x, y, z) = -z^3$ of a non-linearity in equation (1), further investigating an equation

$$\dot{x} + x = p \cdot \dot{x}(t - T) - \dot{x}(t - T)^3. \qquad (5)$$

Note that equation (1) with $F = Az^3$ can be reduced to equation (5) by substitution $x \mapsto x/\sqrt{-A}$ for negative A.

We observe that equation (5) for subcritical values of parameter p, which is $|p| > 1$, has many stable periodic solutions, the periods of which are related to the corresponding imaginary values of roots λ_n to the equation (3), described in Theorem 2.

We study those periodic solutions numerically, obtaining regions in parameter space, in which there are periodic solutions of corresponding periods. We study parameter dependence of the shape and amplitude of those periodic solutions, providing a method of switching between those multi-stable solutions using an external control.

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